

PEANO'S CONCEPTION OF A SINGLE INFINITE CARDINALITY

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Although Peano's negative attitude toward infinitesimals—particularly, geometric infinitesimals—is widely documented, his conception of a single infinite cardinality and, more generally, his views on the infinite are less known. In this article, we reconstruct the evolution of Peano's ideas on these questions and formulate several hypotheses about their underlying motivations.

1. Introduction

Peano is widely known for his contributions to the creation of modern logical symbolism and to the axiomatization of arithmetic—in particular, for the axioms that still bear his name.¹ His proof of the impossibility of infinitesimal segments in geometry has also received due attention (see Peano 1892a).² Less explored, if at all, are Peano's views on the infinite. Precisely when Peano was most active mathematically, this area experienced major and unprecedented breakthroughs,

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1. For an overview of Peano's contributions to logic, see Kennedy (1980), Borga et al. (1985), Grattan-Guinness (2000), and the 2021 special issue of *Philosophia Scientiae*, "The Peano School," edited by P. Cantù and E. Luciano.

2. For a reconstruction of Peano's views about the infinitesimals, see Ehrlich (2006) and the more recent Freguglia (2021).

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thanks especially to the work of Cantor, Dedekind, Schröder, Veronese, and other mathematicians (cf. Grattan-Guinness 2000; Ferreirós 2010).

Peano's early conception of a single infinite cardinality has been scrutinized by Mancosu (2016), who, besides showing where the concept stands in the history of the infinite, also casts "Peano's Principle" as a bona fide cardinality principle able to put pressure on the neologicist doctrine of the analyticity of Hume's Principle.³

Mancosu (2016, 165–66, esp. n. 21) lists the different stages of the evolution of Peano's concept and demonstrates that Peano's ideas became increasingly close to and then, from a certain point onward, indistinguishable from Cantor's. However, he does not pin down the exact motivations behind the concept itself or explain why Peano ultimately endorsed Cantor's transfinite.

The purpose of this article is to fill in the gaps by providing, as far as possible based on extant textual sources, a more articulated account of the evolution of Peano's ideas. This approach might also shed light on the potential significance of Peano's early conception.

With respect to the latter point, our interpretation of Peano's doctrine seeks to establish connections between Peano's and Galileo's conceptions. Galileo's argument in *Dialogues Concerning Two New Sciences* (Galilei 1638) was that the tension between the demands of Euclid's Axiom (the Part-Whole Principle) and those of the "bijection method"—which Galileo himself discusses and uses in a positive way, maybe for the first time in the history of mathematics, to establish the *equinumerosity* of the natural and square numbers—cannot be overcome.⁴ Consequently, Galileo concludes that one should surrender to the idea that there can be no measure of actually infinite collections, as there is of finite ones.

That Peano had quandaries comparable to Galileo's may be deduced from both his lukewarm adherence to (and probably incomplete understanding of) the Cantorian theory and his reliance on other kinds of infinitary intuitions of a geometric character, such as those involved in the construction of the Peano curve. Even when his understanding of the Cantorian theory is adequate, he seems to want to use the theory in a way that best suits his own purposes and "Galilean" conception.

3. Hume's Principle [HP], taken by (Frege 1884) to be the correct statement of 'numerical identity' and, consequently, of the concept of number, is the universal closure of the (second-order) principle $\#x: F(x) = \#x: G(x) \leftrightarrow F \approx G$: "The number of F s is equal to the number of G s if and only if F is equinumerous with G (i.e., if F s and G s can be put in a one-to-one correspondence)." Mancosu (2016, chap. 4) has called the formalization, in second-order logic, of Peano's conception of a single infinite cardinality "Peano's Principle"—that is, $\#x: F(x) = \#x: G(x) \leftrightarrow (\neg Fin(F) \wedge \neg Fin(G)) \vee ((Fin(F) \wedge Fin(G)) \wedge F \approx G)$, "The number of F s is equal to the number of G s if and only if either F and G are infinite or, if finite, equinumerous." Peano's Principle is then discussed in connection with the "good company problem" for HP.

4. Both Euclid's Axiom and the bijection method are discussed in depth in sec. 4.1.

The structure of the article is as follows. In the next section, we make some preliminary considerations; in section 3, we examine the evolution of Peano's ideas; and, finally, in section 4, we consider three possible motivations behind Peano's conception.

2. Did Peano Have a Definite Conception of the Infinite?

From reading Peano's terse statements on the subject, one wonders whether the Italian mathematician had any determinate conception of the infinite at all. Sometimes Peano's ideas seem insufficiently developed to be seen as advocating any specific view.

In some authors' judgment, Peano was brilliant at spotting inaccuracies and improving on already established results but was poorly armed to devise (or just propose) a new theory and rarely took enough care of exposing in full detail, let alone justifying, concepts that he happened to champion.⁵ However, some evidence suggests that this might not be the case regarding the question of the infinite.

To begin with, Peano actively engaged in the lively debates on the nascent theory of sets and on infinitesimals and relentlessly discussed aspects of these topics with Cantor, Russell, Frege, Veronese, and other mathematicians.⁶ Moreover, as mentioned, Peano forcefully opposed Veronese's (geometric) infinitesimals by producing a (purported) proof of their inconsistency (see n. 2 and sec. 4.3). This fact, considered on its own, demonstrates that he was at least willing to go to great lengths to understand questions about the nature of the infinite, both philosophically and mathematically.

Peano's ideas gradually became more akin to Cantor's, to the point that no traces of his earlier conception can be found in later work. What motivated such a noticeable change of mind? We believe that Peano became convinced of the intrinsic weakness of the alternatives to Cantor's conception, including his own, precisely as a consequence of his engagement in the foundational debates that were hosted by, among other journals, his *Rivista di matematica*. The soundness of this interpretation is corroborated by the content of Peano's correspondence with Cantor (1895), which, in our view, proved instrumental for

5. See, e.g., Grattan-Guinness (2000) and Agazzi's introduction to Borga et al. (1985, 7). For a different assessment of the significance of Peano's work, see Rodriguez-Consuegra (1991), esp. the beginning of chap. 3.

6. Traces of such discussions may be found in, among other works, the many letters that Peano exchanged with his mathematical interlocutors, including Cantor (discussed in sec. 3.2). Almost all of Cantor's letters to Peano may be found in his *Briefe* (Cantor 1990). The correspondence between Peano and other mathematicians is reproduced in *Opera Omnia* (Peano 2008).

Peano’s “conversion” to the Cantorian approach as much as his eventual collaboration with arguably the most set-theoretically minded member of his school, Giulio Vivanti.

Nevertheless, even after inspecting much primary and secondary literature, as we have done for this article, the full motivations behind Peano’s early conception and the shift to the Cantorian conception still appear mysterious to some extent. However, the interpretative hypotheses we formulate in section 4 provide some clues.

3. The Evolution of Peano’s Conception

In this section, we closely follow the evolution of Peano’s ideas between 1891 and the turn of the century. For this, we avail ourselves of three fundamental textual sources: (1) Peano’s own articles—in particular, “Sul concetto di numero” (Peano 1891), in which his early conception of the infinite is first formulated, and “Dimostrazione dell’impossibilità di segmenti infinitesimi costanti” (Peano 1892a); (2) Peano’s 1895 correspondence with Cantor, consisting of six letters from Cantor to Peano; and (3) the first three volumes (and sections thereof) of his *Formulaire de mathématiques*, published in rapid succession between 1895 and 1901 (Peano 1895, 1897, 1898, 1899, 1901), in which both Peano’s early and “Cantorian” conception may be found.⁷ We close this section by briefly discussing Vivanti’s role in the evolution of Peano’s ideas.

3.1. “On the Concept of Number”

In 1891, Peano published his famous article “Sul concetto di numero” (On the concept of number) in the *Rivista di matematica*, the journal he had founded the same year. Two years after *Arithmetices principia, nova methodo exposita* (Peano 1889), in which Peano first presented his axioms of the natural numbers, he simplified his notation, demonstrated that his famous five postulates of the natural numbers are mutually independent, and built a more general axiomatic system of numbers by also taking into account relative, rational, and real numbers (cf. Borge et al. [1985, 79–94]; Rodriguez-Consuegra [1991, chap. 3.2]).

“Sul concetto di numero” (Peano 1891) is Peano’s first endeavor to deal explicitly with the infinite. In section 9, he defines a function, *num a*, whose domain consists of “classes” (denoted $a, b, c, \dots u$), and whose values are the “cardinalities”

7. Peano’s letters to Cantor are no longer available, as confirmed in a private email to the authors (February 23, 2022) by Clara Silvia Roero, who recently reedited Peano’s works and correspondence (cf. Peano 2008).

of these classes; in Peano's own words, *num a* is "the number of elements of the class *a*."⁸

Now, if *a* is a finite class, then *num a* is just the (finite) number of its elements—that is, a natural number *n*. But then, Peano states that *num a* is not always a natural number, as the set of natural numbers does not include zero and infinity.⁹ This is the first time Peano mentions the possibility that a class *a* be *empty*, or *infinite* (i.e., that $\text{num } a = \infty$).¹⁰

We wish to say something more substantial about ∞ . Peano (1891) seems to take it to be a bona fide "infinite quantity" that can be manipulated like any other (finite) quantity, as is clear from the propositions 3 and 4 in his section 9. Proposition 3 states, "If *a* and *b* are two nonempty and finite classes having no element in common, then the number of elements of the set of the two classes *a* and *b* is equal to the sum of the number of *as* and *bs*."¹¹ Peano is stating, in modern set-theoretic notation, that if two sets *a* and *b* are disjoint, then the number of the elements of *a* and *b* is equal to the number of the elements of $a \cup b$. Peano then notes that the proposition holds even if one of the two classes, or even both, contain infinite elements. But now we have the following statement as a consequence:

$$x + \infty = \infty + x = \infty \quad (1)$$

where *x* is a finite quantity, and

$$\infty + \infty = \infty. \quad (2)$$

Immediately afterward, he says in proposition 4, "If the classes *a* and *b* are such that the second is contained in the first, and the class *b* is nonempty and is not equal to *a*, and if the number of *as* is finite, then the number of *bs* is also finite and is less than the number of *as*."¹²

8. "Con *num a* intenderemo 'il numero degli individui della classe *a*'" (Peano 1891; 1959, 100). The English translations of Peano's quotes are all ours.

9. "Data una classe *a* non sempre $\text{num}(a)$ è un *N*, poiché *N* non comprende né lo zero, né l'infinito" (Peano 1891; 1959, 101).

10. "Il segno *num* è un segno d'operazione che ad ogni classe fa corrispondere o un *N*, o lo 0, o l' ∞ " (Peano 1959, 102).

11. "Essendo *a* e *b* due classi non nulle e finite, non aventi alcun individuo comune, allora il numero degli individui appartenenti all'insieme delle due classi *a* e *b* vale la somma dei numeri degli *a* e dei *b*" (Peano 1891; 1959, 101).

12. "Se delle classi *a* e *b* la seconda è contenuta nella prima, e la classe *b* non è nulla, e non è eguale ad *a*, e se il numero degli *a* è finito, allora anche il numero dei *b* è finito, ed è minore del numero degli *a*" (Peano 1891).

Then he observes that “this proposition ceases to be valid if $\text{num } a = \infty$.”¹³ Propositions 3 and 4 also demonstrate that ∞ is taken by Peano to be different from Cantor’s ω because, as is known by Cantor’s conception, $\omega + n \neq n + \omega$. However, on the grounds of the arithmetical laws outlined in equations (1) and (2), one could be tempted to view ∞ as being equivalent to \aleph_0 . But this would be a hasty conclusion. In section 4.3, we see that Peano’s own interpretation of his infinitary numbers does not automatically sanction equivalence between his ∞ and \aleph_0 .

One further comment is in order. Proposition 4 states that infinite classes could contain *proper* infinite subclasses. This implies, among other things, that “standard” part-whole relationships cease to be valid in the infinite, which, in turn, may have had some bearings on the development of Peano’s ideas.

3.2. The Cantor-Peano Correspondence

As mentioned, Peano’s early conception of the infinite was gradually modified, and then, by 1899, it was definitively replaced by Cantor’s theory of the transfinite. What ultimately led Peano to change his mind remains an open question. Although the evidence is insufficient, we conjecture that the correspondence between Peano and Cantor in 1895 proved instrumental in that respect.

In what follows, we review salient parts of the discussion Peano entertained with Cantor that are relevant to our purposes. As noted, one side of the correspondence (from Peano to Cantor) is not extant, so Peano’s comments and answers can be deduced only approximately from the responses of his German colleague.¹⁴ Cantor’s letters cover the following subjects: (1) Cantor’s articles that Peano intends to publish in his *Rivista Matematica*, (2) Cantor’s distaste with Veronese’s theory of actual infinitesimals, and (3) Peano’s request for explanations about Cantor’s elusive definition of “cardinal number.” In our view, subjects 2 and 3 are fundamental for the evolution of Peano’s ideas on the infinite. Next we examine them in detail.

In two letters of, respectively, July 27 and July 28, 1895, the discussion focuses on the conflict between Cantor’s and Veronese’s theories. In the German mathematician’s view, what Veronese called “ordered groups” were just plagiarism of Cantor’s “simply ordered sets.” But then Cantor points out what he thinks is the mistake that Veronese has made in trying to automatically extend

13. “Questa proposizione cessa di esistere se $\text{num } a = \infty$ ” (Peano 1891, in Peano 1959, 101).

14. Cantor’s letters to Peano examined in the present work are all collected in the Meschkowski edition of Cantor’s letters (Cantor 1990, 359 ff., nn.143–47). An overview of the contents of the Cantor-Peano correspondence is in Kennedy (1980, 87–90).

the arithmetic of natural numbers to infinite numbers. Cantor observes in two passages of the July 27 letter, “But if it is correct that his $\infty_1 = \omega + {}^*\omega$, his assertion that $2 \cdot \infty_1 = \infty_1 \cdot 2$ must be *incorrect!* . . . Anyway, his [Veronese’s] ‘infinite numbers’ seem tenable to me only if they are identified with some of my ‘transfinite order-types’. In this case, however, they lack the law of commutativity for addition and multiplication (in general) on which he [Veronese] lays so much stress.”¹⁵ We do not know the content of Peano’s answer; in any case, on Cantor’s own impulse (letter of July 28, 1895), the July 27 letter was published by Peano in the August issue of the *Rivista di matematica*. Thus, it seems possible that Cantor managed to convince Peano of the correctness of his arguments.

In subsequent letters, the conversation between the two mathematicians considers Peano’s qualms about Cantor’s definition of “cardinal number.” This subject can be safely deduced from Cantor’s September 14, 1895, letter, in which Cantor tries to clarify some notions contained in his article, “Beiträge zur Begründung der transfiniten Mengenlehre, 1” (Cantor 1895, chap. 5). From what Cantor says, it is clear that Peano was unsure about Cantor’s notion of “finite cardinal number” and about how Cantor introduced the induction principle, and Peano asked his colleague for clarifications.

In his letter of September 21, 1895, Cantor quotes a sentence written by Peano in his response to Cantor’s previous letter: “Where can one find the definition of finite cardinal numbers?”¹⁶ This note is a clear sign that Cantor’s previous letter was not entirely clarificatory or, in any case, that Peano found Cantor’s definitions unconvincing.¹⁷

What seems likely is that, as a consequence of the clarifications Cantor gave Peano (and possibly Cantor’s forceful rejection of Veronese’s theory), Peano started considering Cantor’s transfinite as the only correct conception of the infinite and Cantor’s theory of cardinal numbers as definitive.¹⁸ If he had chosen not to fully adhere to Cantor’s theory in 1891, or simply not to explore it in depth, he now had further material at hand, including Cantor’s clarificatory statements, which could reorient his views. This shift would soon reflect on the *Formulaire*, where the theory of the transfinite is gradually prioritized.

15. “Jedenfalls scheinen mir seine “unendlichen Zahlen” nur dann haltbar, wenn sie mit gewissen von meinen “transfiniten Ordnungstypen” identificirt werden. In diesem Falle fehlt ihnen aber das Gesetz der Commutabilität bei der Addition und Multiplication (im Allgemeinen), worauf er solchen Nachdruck legt” (Cantor 1990, 360). The English translation is ours.

16. “Où est ce que l’on trouve la definition der endlichen Cardinalzahlen?” (Cantor 1990, 365).

17. We thank an anonymous reviewer for drawing our attention to this possibility.

18. However, Peano (1892a, 1892b) had already independently attempted to dash Veronese’s conception (but see our comments in sec. 4.3). For Peano’s assessment of Veronese’s theory, see Fisher (1994) and Cantù (1999).

3.3. The Infinite in the *Formulaire*

We turn to survey, briefly, the modifications of Peano's conception as can be found in Peano's *Formulaire de mathématiques*.¹⁹ As noted, these volumes have already been briefly taken into account by Paolo Mancosu (2016, 165–66, esp. n. 21). We aim to expand on and complement Mancosu's account.

The evolution of Peano's ideas on the infinite in the *Formulaire* spans a period of 4 years and ends in 1899, when all traces of Peano's early conception disappear, and Cantor's transfinite ultimately takes over. The first volume (Peano 1895) saw the collaboration of other Italian mathematicians, such as G. Vailati, F. Castellano, G. Vivanti, and R. Bettazzi. Of particular importance for our purposes are sections V and VI, written, respectively, by Peano and Vivanti; among other things, the latter had taken part in the debate with Bettazzi on the infinitesimals hosted by the *Rivista* in the years 1891–92.²⁰

In section V (Peano 1895), entitled “Classes de nombres,” we find again the mention of just one infinite cardinality, ∞ , which is now explicitly defined as one of the possible values of *num* u , where u is a class, as follows:²¹

$$4. \textit{num } u = \infty \leftrightarrow \textit{num } u \notin \mathbb{N}_0.$$

Then Peano explains that *num* u can take only three values:

$$5. \textit{num } u = n \in \mathbb{N} \vee 0 \vee \infty.$$

Moreover, in definition 6, ∞ is again characterized as enjoying *commutativity*:

$$6. a \in \mathbb{N}_0, \quad a + \infty = \infty + a = \infty + \infty = \infty, \quad \text{and} \quad a < \infty.$$

In contrast, section VI, by Vivanti rather than Peano, introduces and uses the symbols *Nc* and *Ntransf*, denoting, respectively, Cantor's cardinals and ordinals and deals with the corresponding set-theoretic notions. Therefore, in 1895, Peano did not entertain ideas much different from those that appeared in 1891—namely, he did not subscribe to Cantor's views—whereas section VI (Peano 1895), which was composed by Vivanti and mentions Cantor's cardinals and ordinals, has no immediate connection with the one composed by Peano. This is consistent with our hypothesis that the correspondence with Cantor

19. All the volumes of the *Formulaire* appeared as supplements to issues of the *Rivista di matematica* between 1895 and 1908. For an overview of the plan and evolution of this work, see the exhaustive Cassina (1955).

20. The debate is accurately reconstructed by Ehrlich (2006, 75–101).

21. In the text of definitions 4–6, we have, for the sake of simplicity, chosen not to adopt Peano's original notation but the modern one.

may have led Peano to change his mind after the publication of the first volume of the *Formulaire*, which, in any case, must have been drafted well before 1895.

The second volume of the *Formulaire* consists of three parts, each published 1 year apart: section 1, *Logique mathématique* (Peano 1897); section 2, *Arithmétique* (Peano 1898), and section 3, untitled (Peano 1899). Section 2 (Peano 1898) contains the first manifestation of Cantor's theory in Peano's writings:²²

Here we define another idea, indicated by the sign Nc' similar to the previous one, indicated by num , but not identical. This definition is expressed by the only signs of logic (§1); it is therefore independent of the primitive ideas \mathbb{N}_0 , $+$, 0 . We could start arithmetic here. P211 expresses the coincidence of the signs num and Nc' when one deals with finite classes. But we will have for example: $num \mathbb{N} = num \mathbb{R}^+$, because the classes \mathbb{N} and \mathbb{R}^+ are both infinite; but $Nc' \mathbb{N} < Nc' \mathbb{R}^+$, because the power of \mathbb{N} is less than the power of \mathbb{R}^+ . Mr. Cantor indicated the power of a by \bar{a} (see RdM. A. 1895 p.130 [Peano (1895, 130)]), a notation that cannot be adopted in the *Formulaire*. M. Vivanti in F_1 VII §2 P1 [Peano (1895, chap. VI)] has replaced it with $Nc' a$ "the cardinal number of a ."²³

From this quote, it seems clear that Peano is beginning to change his mind and gradually converting to Cantor's theory, although he keeps mentioning both concepts: Nc' , denoting Cantor's cardinal numbers, now appears alongside num as a symbol of "cardinality."

The second volume of the *Formulaire* marks one further transformation of Peano's (1899) conception. In that work, for the first time, Peano "merges" the symbols num and Nc' into a unique, new symbol, Num . Num now denotes Cantor's et cetera, nothing but Cantor's transfinite cardinalities, an "expansion" of Peano's former num subtly revealed by the change of notation. From this point onward, Peano seems to fully adhere to Cantor's theory.

22. In the original text, Peano uses the symbols \mathbb{N} and \mathbb{Q} , to denote, respectively, the set of the natural and the positive real numbers; for simplicity in our translation, we have replaced \mathbb{N} and \mathbb{Q} with the modern \mathbb{N} and \mathbb{R}^+ .

23. "Nous définissons ici une autre idée, indiquée par le signe ' Nc' ' semblable à la précédente, indiquée par ' num ', mais non identique. Cette définition est exprimée par les seules signes de logique (§1); elle est donc indépendante des idées primitives \mathbb{N}_0 , $+$, 0 . On pourrait commencer ici l'Arithmétique. La P211 exprime la coincidence des signes num et Nc' lorsqu'il s'agit de classes finies. Mais on aura par exemple: $num \mathbb{N} = num \mathbb{R}^+$, car les classes \mathbb{N} et \mathbb{R}^+ sont toutes les deux infinies; mais $Nc' \mathbb{N} < Nc' \mathbb{R}^+$, car la puissance des \mathbb{N} est plus petite que la puissance des \mathbb{R}^+ . M. Cantor a indiqué la puissance de a par \bar{a} (Cfr. RdM. a. 1895 p.130), notation qu'on ne peut pas adopter dans le *Formulaire*. M. Vivanti dans F_1 VII §2 P1 l'a remplacée par $Nc' a$ 'le nombre cardinal des a '" (Peano 1898, 39).

In the *Formulaire*'s third volume, in the *Num* section, Peano (1901) finally explains, "*Num'Cls* means 'the number of a class.' These numbers coincide with the [elements of] \mathbb{N}_0 for the finite classes; G. Cantor calls them 'cardinal numbers.' In F 1895 the symbol '*Nc*' was introduced to represent them [again, a reference to Vivanti's section (Peano 1895, chap. VI)]."²⁴

Already by 1901, and then in the subsequent volumes of the *Formulaire* (Peano 1903, 1908), any significant distinction between Peano's and Cantor's conceptions has disappeared, and Peano's early conception of the infinite is only a distant memory.

3.4. Enter Vivanti

We have seen that Giulio Vivanti was involved in the writing of the *Formulaire*, and we have cursorily mentioned that he interacted with Bettazzi on the issue of the existence of actual infinitesimals (see n. 20). His role with respect to the evolution of Peano's ideas might have been equally prominent. Indeed, one could even conjecture that the transformation of Peano's ideas was, directly or indirectly, also due to Vivanti's advice.

This claim seems to be supported by at least two main reasons. The first is that Vivanti was acknowledged to be, at the time, one of the main experts of (Cantorian) set theory. Herbert Meschkowski sharply expresses this fact in a note following Cantor's December 3, 1885, letter to Vivanti: "The number of mathematicians who around 1885 had confronted Cantor's doctrine was still very small; the young Italian Vivanti belonged to them. . . . Vivanti's works on set theory . . . had, moreover, contributed to make Cantor's theory known in Italy" (Cantor 1990, 251; our translation). Indeed, as explained by Meschkowski, Vivanti and Cantor corresponded with each other for a decade (1885–95). As a main correspondent of Cantor's and early practitioner of Cantorian set theory, Vivanti must have realized the potential and overriding strength of the Cantorian theory compared with other conceptions of the infinite.

In the context of the debate on the infinitesimals that appeared in the *Rivista di matematica*, Vivanti had significantly stood out as an advocate of Cantor's conception against Bettazzi. At the same time, Vivanti (1891, 1895) never held the view that theories of infinitesimals were, in the least, inconsistent.

In their exchange on the transfinite, Vivanti and Cantor dealt with a broad range of themes, spanning mathematical and philosophical aspects of set theory,

24. "*Num'Cls* signifie 'le nombre d'une classe'. Ces nombres coïncident avec les N_0 pour les classes finies; G. Cantor les appelle 'nombres cardinaux'. Dans F1895 on a introduit le symbole '*Nc*' pour les représenter" (Peano 1901, 70).

as, for instance, in Cantor's 1886 letter, which would subsequently become part of Cantor's *Mitteilungen zur Lehre vom Transfiniten* (Cantor 1887/1932, 409–11).

In 1893, Vivanti insisted with Cantor that the latter's attempts to expel actual infinitesimals from mathematics were doomed to failure, as he argued that du Bois-Reymond's use of infinitely large and small quantities was perfectly consistent with the concept of number. Cantor was equally harsh in rebutting Vivanti's comments.²⁵

The fruits of Vivanti's lasting collaboration with Peano included an early "Teoria di gruppi di punti," a work on point sets whose publication preceded that of the *Formulaire's* first edition. Moreover, Vivanti had already reviewed "Dimostrazione dell'impossibilità di segmenti infinitesimi costanti" (Peano 1892a) after its publication and, using his wide expertise in set theory, corrected Peano on a few points (see sec. 4.3). Subsequently, Vivanti was entrusted by Peano with the writing of the section on the "Teoria degli insiemi" (Set theory) for the 1895 version of the *Formulaire*.

It is also worth mentioning that Vivanti actively fostered the exchange of letters between Cantor and Peano, as can be gleaned, for instance, from the 1893 letter from Cantor to Vivanti that would be published in the *Rivista di matematica* (see Cantor 1990, 505). The correspondence between Peano and Cantor, as we saw, started two years later in 1895 (Kennedy 1980, 87–90).

In summary, it is plausible to conjecture that Peano's main "set-theoretic" collaborator, Vivanti, sought to steer Peano's ideas, so to speak, toward the set-theoretic conception. In particular, Vivanti may not only have been the main connection between Peano and Cantor but also between Peano and a (more) correct understanding of set theory.

4. Interpretations of Peano's Conception

In this section, we turn to examine the issue of what may have motivated Peano's early conception of the infinite and discuss further textual sources that might help us better understand the evolution of Peano's ideas.

We consider three main potential motivations in the next three sections, respectively. First, like Galileo before him, Peano was not fully convinced of the correctness of the "bijection method" vis-à-vis Euclid's Axiom (the Part-Whole Principle). Second, also of a markedly Galilean character, Peano's work on space-filling

25. See Cantor's December 13, 1893, letter to Vivanti (facsimile), reproduced in Cantor (1990, 514 ff.).

curves may have suggested to him that Cantor's notion of *cardinality* was not suitable for all mathematical contexts. A third possible motivation is that Peano assumed that (the bulk of) Cantor's theory of the transfinite could be used to suit his own purposes or even that Cantor's theory was consistent with his own conception.

4.1. Between Galileo and Euclid

In his *Dialogues Concerning Two New Sciences* (1638), Galileo came to express skepticism about the possibility of comparing the sizes of infinite "collections."

In the work's *First Day*, Galileo's spokesperson, Salviati, proposes to compare the sizes of infinite collections of objects by checking that each object in one collection is uniquely matched by an object in the other collection. Specifically, Salviati proposes to compare square with natural numbers through a "mapping function," $n \mapsto n^2$, which associates to each natural number its square number:

$$\begin{aligned} 0 &\leftrightarrow 0 \\ 1 &\leftrightarrow 1 \\ 2 &\leftrightarrow 4 \\ 3 &\leftrightarrow 9 \\ .. &\leftrightarrow .. \end{aligned}$$

Salviati and his interlocutor, Sagredo, agree that, using this method, it must be concluded that there are as many natural numbers as square numbers—that $\mathfrak{s}(\mathbb{N}) = \mathfrak{s}(\mathbb{S})$, where $\mathfrak{s}(X)$ means "size of X ." But now the dire puzzle unfolds in front of their eyes: it seems to be an "established fact," common knowledge, that there are more natural than square numbers, as \mathbb{S} has gaps "in between" that \mathbb{N} does not have; this is natural, as \mathbb{S} is a proper part of \mathbb{N} (i.e., using the set-theoretic notation, $\mathbb{S} \subset \mathbb{N}$). Relying on intuitions referring to the "density" of \mathbb{S} in \mathbb{N} , Salviati suggests that one should correctly conclude that $\mathfrak{s}(\mathbb{N}) > \mathfrak{s}(\mathbb{S})$. This is what has come to be known as Galileo's Paradox.

Salviati asks, Which of the two horns of the dilemma is correct? Neither, apparently, as Galileo ultimately pointed out: "This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another" (Galilei 1638, 31). Galileo does not go so far as to say that, but as one cannot refer to one size as being greater than, less than, or equal to another in the infinite, it would, in principle, be legitimate and consequential to think that there is but *one* infinite cardinality, for all intents and purposes. Thus,

what we would have at hand (Galilei 1638) would be the enunciation of an early “single-cardinality” conception of the infinite dictated by the impossibility to come to terms with a conflict between two ways of counting in the infinite, one based on “bijections” and one on “densities.” The former method is formally enshrined in what would later become the central pillar of set theory:

Cantor’s Principle. Given two sets A and B , if there exists an injective and surjective $f : A \rightarrow B$, which is 1-1 and onto (i.e., a bijection between A and B), then $\aleph(A) = \aleph(B)$.²⁶

The latter method upholds what was already known in antiquity as Euclid’s Axiom (Common Notion V of the *Elements*):

Part-Whole Principle. Given two sets A and B , if $A \subset B$, then $\aleph(A) < \aleph(B)$.

In our era—the “set-theoretic era,” as it were—one tends to think straight away that Cantor’s Principle is the correct way to measure infinite collections and that, consequently, Galileo’s Paradox is no paradox at all but rather the hallmark of the infinite itself, as, in particular, expressed by:

Dedekind Infiniteness. A set is *infinite* if and only if it can be put in a one-to-one correspondence with a proper part of itself; otherwise, it is finite.²⁷

Nevertheless, the correctness of the Part-Whole Principle, as is known, was the prevailing view for many centuries and still lay almost uncontested in Galileo’s time.²⁸ Moreover, oscillations between compliance with Cantor’s Principle and the Part-Whole Principle could even be found, centuries later, in the work of avowed supporters of the bijection method like Bolzano (see Mancosu 2016, 130–31).

One could speculate (1) that Peano shared Galileo’s concerns about the possibility of articulating a theory of *distinct* infinite cardinalities and (2) that he settled on a sober “one-cardinality” conception precisely because he was hesitant to choose among one of the two (known) methods of counting in the infinite.

26. Cantor’s Principle may also be seen as a “variant” of Hume’s Principle, mentioned in sec. 1 (see n. 3).

27. The notion was first formulated by Dedekind (1888, V, 64).

28. For a careful excursus of the history of several cardinality principles, we refer the reader to Mancosu (2009).

In particular, Peano's 1891 conception might have reflected dissatisfaction both with a purely Cantorian and a Euclidean point of view. This is (indirectly) proved by a remark made by Peano (1891) in a footnote concerning Rodolfo Bettazzi's (1887) use of the bijection method to characterize the concept of number.²⁹ Bettazzi thought that if two sets A and B could be put in a one-to-one correspondence, then any correspondence between them would have to be one-to-one. For the sake of our exposition, let us restate this:

Bettazzi's Principle. Given two sets A and B , if there exists a bijective function between A and B , then there is no function between A and B that is injective but not surjective.

Peano correctly objects to Bettazzi that one can biject infinite sets with their own proper infinite parts—that is, one can have *injections* of sets with themselves that, clearly, are not *surjective*. Further reflection on this fact might have led Peano to doubt that the bijection method could yield a satisfactory characterization of the notion of (infinite) number, as envisaged by Bettazzi, and the most natural consequence of this would have been that he would not allow for the existence of different sizes in the infinite.

On the other hand, and regardless of his assessment of Bettazzi's Principle, if Peano had wanted to stick with the Part-Whole Principle, then he would have had to allow for different infinite sizes anyway—for example, that of the *odd* numbers and of \mathbb{N} itself—but this is inconsistent with his conception as we know it.

Overall, Peano's hesitancy between Euclid and Cantor, so to speak, nails down the “Galilean” character of his standpoint, which could be summarized as follows: there is no way to provide scope for different infinite sizes in a way that is consistent with both the Cantorian and Euclidean ways of counting in the infinite; consequently, one has no other choice but to prudently retreat to a single-cardinality conception of the infinite.

4.2. Equinumerosity and Continuity

Another reason for skepticism about the “measurability” of the infinite could have potentially been suggested to Peano by his work on space-filling curves, such as his own curve (fig. 1). We discuss how in more detail.

Cantor (1878) discussed groundbreaking results on bijective correspondences between points sets in different *dimensions*. In particular, Cantor proved the bijectability of the closed interval $I = [0, 1]$ with the unit square

29. The passage is cited (and translated to English) by Mancosu (2016, 164–65).

4.3. (Geometric) Infinitesimals and Cantorian Set Theory

Peano (1892a) attempted to improve on Cantor’s earlier proof that actual infinitesimal segments are inconsistent.³¹ In the article, Peano shows a more than decent understanding of Cantorian set theory but also seems to want to use it originally, in a way that is compatible with his single-cardinality conception of the infinite.

Key notions in that work are those of “bounded” and “infinitesimal” segments. Given the half-line with origin in o , a *bounded* segment is a segment op with origin in o and end in p . Peano also denotes a bounded segment with u , or with \overline{ou} . An *infinitesimal* segment is an \overline{ou} lying inside a bounded segment v denoted as $u = v/\infty$. The full definition of u reads as follows:

We say that the *segment* u is infinitesimal with respect to v and we write $u \in v/\infty$, if every multiple of u is less than v .³²

Then Peano proceeds to define the multiple of infinite order of u , “ ∞u .” He says:

We shall posit:

$$\infty u = \bigcup \mathbb{N}u$$

that is, we call multiple of u of infinite order the set of points, which either lie on some segment $u, 2u, 3u \dots$ or the upper bound of the multiples of u .³³

Peano asserts that by the definition of “infinitesimal,” if u is infinitesimal, then ∞u must lie inside v and so must all other infinitary multiples of u . In a crucial passage, Peano (1892a) says, “We can add ∞ to itself, thus obtaining $2\infty u$, and, generally, we can form all multiples of ∞u ; we can multiply ∞u by ∞ , and obtain $\infty^2 u$ and so on. But all these various segments, which one obtains multiplying u by Cantor’s transfinite numbers, are all equal to one another.”³⁴ Consequently,

31. A detailed analysis of Cantor’s (1887/1932) proof is carried out by Ehrlich (2006, 27–51).

32. “Dicesi che il segmento u è infinitesimo rispetto al segmento v , e scriveremo $u \in v/\infty$, se ogni multiplo di u è minore di v ” (Peano 1892a; 1959, 113). Note the use of Peano’s ϵ symbol, which means “is.”

33. “Porremo $\infty u = \bigcup \mathbb{N}u$ cioè chiamiamo multiplo d’ordine infinito di u l’insieme dei punti che stanno sopra qualcuno dei segmenti $u, 2u, 3u \dots$ o il limite superiore dei multipli di u ” (Peano 1892a; 1959, 113). For the sake of simplicity, in the formula, we have chosen to replace Peano’s original notation: $\bigcup' \mathbb{N}u$ with $\bigcup \mathbb{N}u$.

34. “Possiamo sommare ∞u con sé stesso, ottenendo così $2\infty u$, ed in generale possiamo formare tutti i multipli di ∞u ; possiamo moltiplicare ∞u per ∞ , ed ottenere $\infty^2 u$ e così via. Ma tutti questi vari segmenti, che si ottengono moltiplicando u pei numeri transfiniti di Cantor sono eguali fra loro” (Peano 1892a; 1959, 113).

infinitesimal segments, if such things existed, would blatantly violate the properties of bounded segments, which require that, for instance, a segment of length $2\infty u$ be greater than ∞u . Peano concludes that infinitesimal segments are inconsistent with “standard” representations of the geometric space.

Leaving aside entirely the issue of the correctness of Peano’s argument, what is most relevant to our purposes is Peano’s peculiar use of Cantor’s transfinite numbers and arithmetic. This is indeed puzzling, as, in Cantor’s theory, $\omega + 1 > \omega$, thus, contrary to what asserted by Peano, $\omega + 1 \cdot u$ is greater than $\omega \cdot u$.

Indeed, Giuseppe Veronese was immediately able to raise precisely this issue. Meant to be a response to Peano (1892a), Veronese (1892) correctly diagnoses what seems to be the trouble with Peano’s proof: “But these equalities [$\infty u = 2\infty u = \infty^2 u = \dots$] do not depend on the properties of Mr. Cantor’s transfinite numbers, for which hold: $\omega + 1 > \omega$, $2\omega > \omega$, etc., but precisely on considering ∞u as *unlimited*” (74; emphasis added). In the passage, Veronese sharply points out that it is Peano’s own definitions of ∞u , $2\infty u$, and so forth, not Cantor’s *transfinite arithmetic*, that allow Peano to assert that these quantities are all equal. Similar remarks were made by Giulio Vivanti (1895), who explained why the argument was doomed to failure. Vivanti argued that Peano viewed $(\infty + 1)u$ as the upper bound of $(n + 1)u$, but now the points of $(\infty + 1)u$ must lie either on one of the finite multiples of u or be $(\infty + 1)u$ itself. So, $(\infty + 1)u$ is precisely the *same* segment as ∞u ; therefore, $(\infty + 1)u$ and ∞u must be *equal*. But, as Vivanti (1895, 69) points out, if it is true that Cantor’s ω is the limit of n , it is also true that $\omega + 1$ is not the *limit* of any number, and, as pointed out by Veronese, $\omega + 1 > \omega$ and $(\omega + 1)u > \omega u$.

Peano’s reliance on Cantor’s transfinite numbers in his proof does not seem to serve well his purpose of showing that geometric infinitesimals are inconsistent. To rescue the force of the argument, Freguglia (2021, 152 ff.) conjectures that the reason why Peano holds that $\infty u = 2\infty u = \infty^2 u = \dots$ is the fact that Peano “assimilates” ∞ to \aleph_0 .³⁵ Freguglia argues that this makes sense because, although $\omega \neq \omega + 1$, on the contrary, $\aleph_0 = \aleph_0 + 1$, and in general, if κ is a transfinite *cardinal* number, and n a natural number:

$$\kappa + n = \kappa \cdot n = \kappa^n = \kappa.$$

Freguglia’s interpretation has some merit but seems to miss the mark. In particular, it seems unjustified to force this interpretation on Peano’s multiple references to Cantor’s transfinite numbers; the numbers that Peano is referring to here are, it

35. Freguglia (2021) says, “Peano explicitly assimilates ∞ to \aleph_0 ” and then explains in n. 14, “in the sense that it has the same arithmetic behavior.”

seems to us and as understood by Veronese and Vivanti, just Cantor's transfinite ordinals ($\omega, \omega + 1, \dots, \omega + \omega$).

Following the spirit but not the letter of Freguglia's interpretation, one could say that Peano may have taken his numbers ($\infty u, \infty^2 u, \dots$) to be equivalent to \aleph_0 in the sense that he thought that they were linearly ordered, like Cantor's ordinals, but that they had, nonetheless, the same cardinality (as they are all countable). By this interpretation, because what counts as length of a geometric segment is the measure expressed by a cardinal number, one must conclude that segments of length $\infty u, \infty^2 u$ cannot but be equal. But there is a problem with this interpretation: Cantor's ordinals may also be uncountable—that is, they may have different cardinalities, and Peano might have been aware of this.

We wish to put forward one last interpretation of Peano's argument, which aligns more with the content of Vivanti's and Veronese's comments above. By this interpretation, Peano viewed his infinite numbers as being all equal to each other not because he thought that ∞ was equivalent to \aleph_0 but because he thought that there was just one infinite cardinality. When Peano refers to "Cantor's numbers" in the aforementioned passage, he is really likening his own numbers to Cantor's ordinals (as supposed by Veronese and Vivanti), but he is also committing himself to the view that those numbers must all have the same cardinality because, by his own conception, there exists only one, ∞ , which cannot be transcended. Eventually, Peano's use of Cantor's numbers in the proof could be seen as an original attempt to merge his own, "single-cardinality" conception with Cantor's transfinite.

In summary, the three motivations we have examined seem to support the following conclusion: At least in the years 1891–95, Peano advocated a carefully thought through and original conception of the infinite, which he believed was motivated by several practical mathematical contexts (see, e.g., Peano 1890, 1892a) and reasons. The fact that he subsequently abandoned this conception does not mean that he thought it to be faulty, "half scarce made up" in any respect, but only that he gradually came to acknowledge that Cantor's theory was more general, far-reaching, and, ultimately, fruitful than his own.

5. Conclusion

We have reviewed the development of Peano's conception of the infinite as far as the publication of the third volume of his *Formulaire* in 1901. We have seen that by the publication of the third section of the second volume in 1899, Peano's conception had aligned with the already mainstream set-theoretic conception. In our view, both Cantor and Vivanti may have played a role in persuading Peano to abandon his earlier view.

As far as the view itself is concerned, it seems to us to have adequately shown that Peano had perfectly clear in his mind its consequences, the way it could be properly used in mathematical contexts, and how it related to and could even merge with aspects of Cantor's transfinite.

Moreover, we have seen that until his full set-theoretic "conversion," Peano might have been doubtful of the efficacy and adequacy of the bijection method, especially insofar as it conflicted with the Part-Whole Principle.

Finally, we have conjectured that these doubts might have led Peano to support a conception of single infinite cardinality that partly incorporates and validates Galileo's skeptical view of the measurability of the actual infinite.

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