Chapter 3
Maddy On The Multiverse

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Abstract Penelope Maddy has recently addressed the set-theoretic multiverse, and expressed reservations on its status and merits (Maddy, Set-theoretic foundations. In: Caicedo et al (eds) Foundations of mathematics. Essays in honor of W. Hugh Woodin's 60th birthday. Contemporary mathematics. American Mathematical Society, Providence, pp. 289–322, 2017). The purpose of the paper is to examine her concerns, by using the interpretative framework of set-theoretic naturalism. I first distinguish three main forms of 'multiversism', and then I proceed to analyse Maddy's concerns. Among other things, I take into account salient aspects of multiverse-related mathematics, in particular, research programmes in set theory for which the use of the multiverse seems to be crucial, and show how one may provide responses to Maddy's concerns based on a careful analysis of 'multiverse practice'.

3.1 The Problem

The development of set theory has progressively brought to the fore the problem of whether set theory should be interpreted as the theory of a *single* universe of sets, V, or whether it should be viewed as a theory about *multiple* structures (universes), that is, about a set-theoretic 'multiverse'.

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The universe/multiverse dichotomy is just an *ontological* (*semantic*) variant of the pluralism/non-pluralism dichotomy. The (proof-theoretic) pluralist believes that there is no unique, or preferred, theory T of sets, and that all theories express, to some extent, some properties of sets. For instance, the pluralist may believe that both ZFC+CH and ZFC+¬CH are equally valid and interesting theories of sets, insofar as they express alternative, but, in principle, equally acceptable, properties of sets. *Multiversism* is pluralism in its ontological/semantic form. It should be noted that one may be both a pluralist *and* a multiversist: that is, one may both believe that there are many, equally valid theories of sets and also that there are many, equally valid set-theoretic structures. Non-pluralism and, at the ontological level, *universism*, hold, respectively, that there is a single correct theory T of sets, or that there is a *unique* structure embodying all set-theoretic truths.[1]

The issue of which, between universism and multiversism, suits best set theory and its goals is crucial both for the philosophy of set theory and, more generally, for the philosophy of mathematics, as arguments in favour of multiversism may be taken to count as arguments in favour of the *absolute undecidability* of some set-theoretic statements, and we have evidence that multiversism is precisely construed in this way by some authors.[2]

The terms of the conceptions sketched above, and their relevance for the philosophy of mathematics, are rather uncontroversial to anyone. What, however, does seem contentious is whether the universe/multiverse dichotomy, and the possible adoption of a specific conception of the set-theoretic multiverse, is relevant to set-theorists and to their mathematical work. We know that set-theorists work with a plurality of models, such as models obtained through forcing, ultrafilter constructions, elementary embeddings, inner models like L, and many others. Now, how does the use of one (or more) collection(s) of set-theoretic models, that is, of the 'multiverse(s)' of a given theory T, contribute to their work? Are there any practical and foundational advantages in taking some collections of models, such as, say, countable transitive models or set-generic extensions of ZFC, as being especially relevant to set-theoretic work, and also as acting as a foundational framework for set theory?

This paper aims to provide answers to these questions, by using a specific, and especially authoritative, point of view, that of Maddian naturalism, and, in particular, by responding to the concerns that, in Maddy (2017), Maddy has expressed about the value and usefulness of a 'multiversist' approach.[3]

[1] I am indebted to Koellner (2014) for this articulation of the pluralist/non-pluralist positions.

[2] This seems to be Väänänen's point of view in Väänänen (2014). Väänänen's main goal is to articulate a position which allows one to express the absolute undecidability of set-theoretic statements which are currently undecidable in several important theories (such as ZFC plus large cardinals). See also Sect. 3.2.1.

[3] It should be noted that the article mentioned is, in fact, an appraisal of *different* competing foundations of mathematics, also including set theory, and of the roles such competing foundations carry out. Only one specific section of the article is explicitly devoted to examining the prospects of the set-theoretic multiverse.

Over the years, Maddian naturalism has progressively, and coherently, come to the fore as one of the most influential positions in the philosophy of set theory, and in general, in the naturalist philosophy of mathematics. The position is generally associated to the following views:[4]

1. Metaphysical issues are irrelevant to the practical development of set theory, as well as to the justification of its internal techniques and results
2. The proper method of a naturalist philosophy of set theory consists in using rational methodologies attentive to intra-set-theoretic practice which altogether rule out extra-mathematical considerations
3. The justification and adoption of set-theoretic principles/axioms is also guided by methodologies of this sort

Especially points 1. and 2. in the summary above are crucial for set-theoretic naturalists: we ought not to evaluate mathematics, its goals and results, using extra-mathematical views or conceptions. The professed ideal of a 'second philosophy', that is of a philosophy of set theory which is especially attentive to *practice*, is the central tenet of the set-theoretic naturalist.[5]

Now, one further view distinctively associated to Maddy's naturalism is that our conception of the universe, V, is justified in light of set-theoretic practice, and in view of our set-theoretic purposes, insofar as set theory is pre-eminently guided by the strive to find a *unifying* account of mathematical phenomena. Therefore, if the 'unification' goal has priority over other goals, we ought to accept V as being the most suitable foundational framework for our set-theoretic investigations. This, in turn, sanctions the view that set theory essentially deals with proving facts about/establishing properties of V, a view which, as is clear, places Maddian naturalism in the universist camp.

In Maddy's works, the set-theoretic goal of unification has progressively taken the shape of the maxim 'unify'. Through fostering 'unify', set theory is thought to have been able to become the unique and far-reaching subject that it is today, and the maxim also serves as a spur to pursue further the search for solutions to the open set-theoretic problems. This point of view is articulated by Maddy as follows:

> If set-theorists were not motivated by a maxim of this sort, there would be no pressure to settle CH, to decide the questions of descriptive set theory, or to choose between alternative axiom candidates; it would be enough to consider a *multitude* [my italics] of alternative set theories. (Maddy 1997, p. 210)

[4]Throughout the paper, I shall use 'Maddian naturalism' and 'set-theoretic naturalism' (sometimes, just 'naturalism') interchangeably. Of course, there are many other ways to spell out naturalism in the philosophy of mathematics. An overview of all such positions is in Paseau (2016).

[5]Maddy herself has put forward and discussed the central aspects of set-theoretic naturalism as a form of 'second philosophy' in several works, starting with Maddy (1996). Second philosophy is further delineated in Maddy (2007), as well as in Maddy (2011).

As we know, the ‘independence phenomenon’ has introduced a rather different picture of the ontology of set theory, one which seems to be more compatible with a pluralist account of set-theoretic phenomena. This poses pressing questions for the Maddian naturalist described above: should she entirely disregard the issue of pluralism or actively engage with it? Moreover, since ‘unification’ concerns seem paramount in her account of set theory, on what alternative grounds may she adopt a pluralist picture? What will be of ‘unify’ within such a picture?

The paper also aims to explore these questions, with a view to providing arguments which may support the following two claims: the universe/multiverse dichotomy is relevant, in many ways, to the naturalist’s approach, and, secondly, the multiverse may be as acceptable as the universe, from a naturalist perspective, for the foundational purposes of set theory.

The structure of the paper is as follows. I first review several conceptions of the set-theoretic multiverse (Sect. 3.2), and provide a classification. I then proceed to summarise Maddy’s concerns (Sect. 3.3), which, overall, will take the shape of five main problems for the multiverse supporter. Finally, in the larger Sect. 3.4, I discuss aspects of the multiverse and of multiverse-related mathematics which seem to adequately respond to the issues raised in Sect. 3.3.

3.2 Multiverse Conceptions

First, we need to clarify what the set-theoretic multiverse consists in, and this may already be a daunting task. Even based on a minimal perusal of the existing literature, it is clear that there is no such thing as *one* mathematical conception of the set-theoretic multiverse, but rather *a bunch of* them, and, in addition, several, alternative research programmes which are variously connected to all such conceptions.

One main difficulty in addressing the set-theoretic multiverse, therefore, is precisely the absence of a shared framework wherein one may discuss results and methodologies. In what follows, I propose a classification of multiverse conceptions, which suits my specific goals. There is nothing compelling about the classification, nor is there any a priori need to classify multiverse conceptions, for that matter.[6]

[6] In Antos et al. (2015), the authors adopt a classification based on the realism/non-realism divide. Hamkins’ conception, for instance, counts as realist, whereas the Hyperuniverse Programme as non-realist. Väänänen proposes a different classification in Väänänen (2014), pp. 191–2: he divides conceptions into *countable* (Hyperuniverse Programme), *full* (Hamkins) and *set-generic* (Woodin, Steel).

3.2.1 Naive Multiversism

It is maybe noteworthy to mention that some form of multiverse thinking was already at work in the characterisation of the universe, since, historically, one could find the first description of a multiverse in Zermelo's characterisation of V. As is known, Zermelo proved the *(quasi-)categoricity* of his system of second-order set theory by showing that V_κ, where κ must be, at least, a strongly inaccessible cardinal, is, up to isomorphism, a model of the axioms ZFC_2 (that is, ZFC with second-order Separation and Replacement).[7] However, since there is an absolutely infinite collection of strongly inaccessible cardinals, one may generate V_κ's of increasing height, a 'tower' of universes, each the rank initial segment of the other. This could, in turn, be seen as a *height multiverse*, in which all universes have the same width (that is, no 'new' subsets may be added), but different heights ('new' ordinals, and ordinal-indexed stages, may be added).[8]

Leaving history aside, the most basic way to express a multiversist attitude is through taking note of the existence of many models of the axioms (of ZFC, for instance). As said at the beginning, we all know that set-theorists work with several kinds of models, through which they may, among other things, prove independence results. 'Naive multiversism' is just the idea that no single model $\mathcal{M}$ of a theory of sets T, should be viewed as 'special', as being *the* universe of sets, *the* collection of all sets.

Saharon Shelah, for instance, has compared models of the axioms to *individuals*, each, presumably, endowed with unique features, but all belonging to the *same* species.[9]

Naive multiversism could also be characterised in a different way, that is, as a conception which incorporates semantic pluralism in an ontologically *monist*

[7] See Zermelo (1930). An examination (and reprise) of Zermelo's proof is in Martin (2001).

[8] For Zermelo's height potentialism, see Linnebo (2013) and Ternullo and Friedman (2016). Of course, historically, Zermelo did not construe the ideas contained in Zermelo (1930) in the current multiversist terms. Väänänen suggests a different reason why Zermelo's characterisation of V ought not to be viewed as an instance of the multiverse. He notes that believing in the existence of an inaccessible κ in V means accepting the axiom: '$\exists\kappa$ inaccessible', which, although clearly independent from ZFC, is not indeterminate in the same sense as, say, CH is. According to Väänänen, the multiverse phenomenon takes place in the presence of statements which are indeterminate in the sense specified. In other terms, if V is V_κ, where κ is the least strongly inaccessible cardinal, then '$\exists\kappa$ inaccessible' is just false, and there is no 'parallel' V with inaccessibles. Cf. Väänänen (2014), p. 187. Finally, one could resist this interpretation of Zermelo's conception by asserting that the set-theoretic hierarchy is, in fact, fully *actual* in height and width. For a fuller examination of the actualism/potentialism dichotomy, see Antos et al. (2015) or Koellner (2009).

[9] Cf. Shelah (2003), p. 211. It is worth quoting the passage in full: 'My mental picture is that we have many possible set theories, all conforming to ZFC. I do not feel "a universe of ZFC" is like "the Sun", it is rather like "a human being" or "a human being of some fixed nationality"'.

framework. Recently, Väänänen seems to have articulated a conception of this sort. He says:

> …we want two universes in order to account for absolute undecidability and at the same time we want to say that both universes are everything. We solve this problem by thinking of the domain of set theory as a multiverse of parallel universes, and letting variables of set theory range intuitively over each parallel universe simultaneously. The axioms of the multiverse are just the usual ZFC axioms and everything that we can say about the multiverse is in harmony with the possibility that there is just one universe. But at the same time the possibility of absolutely undecidable propositions keeps alive the possibility that, in fact, there are several universes. The intuition that this paper is trying to follow is that the parallel universes are more or less close to each other and differ only at the edges. (Väänänen 2014, p. 182)

The main reason for this choice is that:

> ..the intuition about the multiverse is not that everything that is logically possible should also happen in some universe (which would lead to the full multiverse), but that the multiverse is one universe the boundaries of which are verschwommen ("blurred"), as von Neumann wrote. (Väänänen 2014, p. 199)

Väänänen distinguishes set-theoretic statements which are *invariant* for all (potentially alternative, but, in fact, co-existing) V's, and those which are not. ZFC belong to the first class, whereas CH to the second one. 'Multiverse axioms' will capture this intuition: introducing a special logic for independence, and a special symbol for independent statements such as CH ($\neq$CH), one may yield the theory ZFC+ $\neq$CH, which, on the one hand, expresses the truth of ZFC over all of V's, and, on the other, the fact that there are V's which violate CH.[10]

3.2.2 Instrumental Multiversism

A second, more developed, approach to the multiverse may be called 'instrumental multiversism'. By this view, the multiverse is an important mathematical tool, whose philosophical (ontological as well as epistemological) status may not be so relevant.

We may identify the 'multiverse-as-a-tool' conception with two specific, and influential, research programmes in the foundations of set theory. The first is the set-generic multiverse, first addressed by Hugh Woodin, and then taken up and re-formulated by John Steel.[11] The second is the Hyperuniverse Programme (HP), initiated and investigated by Sy Friedman and his collaborators.[12] I start with the latter, which is the form of multiverse I know best.

[10] See Väänänen (2014), pp. 196–202.

[11] For Woodin's set-generic multiverse, see Woodin (2011), although earlier instances of multiverse thinking may also be found, as will be shown later in this paper, in Woodin (2001). Steel's conception of the set-generic multiverse is in Steel (2014).

[12] For this conception, see, essentially, Arrigoni and Friedman (2013), and Antos et al. (2015).

The hyperuniverse, as explained by the authors, is:

> ...the collection of all transitive countable models of ZFC. Within the programme, such models are viewed as a technical tool allowing set-theorists to use the standard model-theoretic and forcing techniques (the Omitting Types Theorem and the existence of generic extensions, respectively). The underlying idea is that the study of members of the hyperuniverse allows one to indirectly examine properties of the real universe V [...]. (Antos et al. 2015, p. 2484)

The hyperuniverse becomes indispensable once one realises that theories T expressing maximality principles which address 'extensions' of V have models if and only if V is countable.[13] The hyperuniverse, therefore, has no significance *per se*, but only as a way to develop the study of: (1) *alternative* maximality principles; (2) first-order consequences of maximality principles (or refinements thereof).

I now proceed to examine the set-generic multiverse. Woodin defines it as the smallest collection of universes $\mathbb{V}_M$ generated by a countable transitive model M of ZFC, such that, if $M \in \mathbb{V}_M$, then also all forcing extensions of M (and generic refinements thereof) are in $\mathbb{V}_M$. It should be noted, already, that Woodin, ultimately, rejects the 'multiverse position', as he shows that, modulo the Ω-conjecture and the assumption of the existence of class-many Woodin cardinals (for which see Sect. 3.4.2.1), reasonably formulated multiverse laws must be violated. However, later on (again, Sect. 3.4.2.1) I will show that multiverse thinking has proved fundamental for many of Woodin's mathematical goals.

Steel characterises the set-generic multiverse as a collection of 'worlds'. He uses a two-sorted language, one sort for sets and one for worlds. Steel's multiverse is modelled upon Woodin's $\mathbb{V}_M$, but there is a crucial difference between the two accounts: Steel's multiverse is expressed by a collection of axioms, which its author calls MV.[14] Now, what matters most to Steel is to 'maximise the interpretative power' of his theory, therefore the MV axioms prescribe that all worlds share, as basic multiverse truths, ZFC and large cardinals (LCs).[15] Where, obviously, worlds do not agree with each other is on the truth-value of independent statements such as CH. Such statements are provisionally deemed not to be 'expressible' in multiverse language, and one of the purposes of Steel's MV theory is precisely to explore the limits of 'expressibility' (and 'meaningfulness') within multiverse language (I shall return to this in Sect. 3.4.4).

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[13] See Sects. 3.4.1 and 3.4.2 later in this paper. Further technical details concerning the use of V-logic and the 'reduction to the hyperuniverse' may be found in Antos et al. (2015) and in Friedman (2016).

[14] Moreover, some of these axioms (such as Axiom 2d [Amalgamation]) are not valid in Woodin's multiverse. Woodin has also tried to axiomatise the generic multiverse, by introducing the aforementioned 'multiverse laws'. However, these laws are so very much *ad hoc* that one would naturally refrain from viewing them as axioms.

[15] This is because, given any two theories T_1 and T_2 extending ZFC having the same consistency strength as ZFC+"exist infinitely many Woodin cardinals", then at least up to Π^1_ω 'statements' (second-order arithmetic), we have that either $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$, and this result may be extended to slightly more complex sentences.

Overall, also the set-generic multiverse exemplifies a practice-oriented attitude to the multiverse: in the same way as HP is guided by the goal of exploring maximality principles in (countable) models satisfying ZFC, Steel's set-generic multiverse aims to capture further set-theoretic truths in all set-generic universes which think ZFC+LCs.

3.2.3 Ontological Multiversism

The last strand of multiversism I describe is very different from all those described so far, and may legitimately be called 'ontological multiversism', that is, the view that the multiverse is a determinate *reality*, consisting of particular *entities*, the models of set theory. Hamkins' broad (radical) multiverse conception is such a form of multiversism.

Hamkins characterises this view as follows:

> ..the fundamental objects of study in set theory have become the models of set theory, and set-theorists move with agility from one model to another. [...] This abundance of set-theoretic possibilities poses a serious difficulty for the universe view, for if one holds that there is a single absolute background concept of set, then one must explain or explain away as imaginary all of the alternative universes that set-theorists seem to have constructed. (Hamkins 2012, p. 418)

As to the issue of what universes there are in the multiverse, Hamkins says:

> The background idea of the multiverse, of course, is that there should be a large collection of universes, each a model of (some kind of) set theory. There seems to be no reason to restrict inclusion only to ZFC models, as we can include models of weaker theories ZF, ZF^-, KP, and so on, perhaps even down to second-order number theory, as this is set-theoretic in a sense. At the same time, there is no reason to consider all universes in the multiverse equally, and we may simply be more interested in the parts of the multiverse consisting of universes satisfying very strong theories, such as ZFC plus large cardinals. (Hamkins 2012, p. 436)

Thus, Hamkins' multiverse is so broad as to include even non-well-founded models, and models of any collection of set- and class-theoretic axioms. Therefore, Hamkins' view is a form both of *proof-theoretic* and *model-theoretic* (*ontological*) pluralism I hinted at in the opening of the paper (p. 1). One peculiar feature of this conception is worth stressing straight away: according to its author, universes in the broad multiverse should be viewed as independent *existents*, just like more 'standard' mathematical entities (numbers, shapes, transfinite ordinals, etc.). The following quote illustrates Hamkins' full avowal of ontological realism (Platonism):

> The multiverse view is one of higher-order realism–Platonism about universes–and I defend it as a realist position asserting actual existence of the alternative set-theoretic universes into which our mathematical tools have allowed us to glimpse. The multiverse view, therefore, does not reduce via proof to a brand of formalism. (Hamkins 2012, p. 417)

Let's take stock. There are at least three conceptions of the set-theoretic multiverse at hand, 'naive', 'instrumental' and 'ontological'. However, 'naive multiversism' does not seem to foster a sufficiently characterised notion of the set-theoretic multiverse. Therefore, in the rest of the paper, I will only be concerned with the other two forms of multiverse.

3.3 Maddy's Assessment of the Multiverse

I now proceed to summarise Maddy's concerns. For the sake of economy, I won't always quote Maddy's text in full, and, in some cases, I will just provide the references to the relevant pages in Maddy (2017).

Maddy notes that set-theoretic naturalists must acknowledge that set theory fulfils specific *foundational* roles, which are consistent with, and originate from, their picture of mathematics as being part of our *best* scientific theory of the world.[16] However, some of these roles are seen by her as spurious, others as appropriate.[17] Among the appropriate ones, Maddy lists 'Shared Standard', 'Generous Arena' and 'Metamathematical Corral'. 'Shared Standard' is the idea that set-theoretic proofs constitute the standard of 'proof in mathematics', whereas 'Metamathematical Corral' refers to the role played by set theory in allowing mathematicians to carry out metamathematical investigations, such as the search for consistency proofs.

'Generous Arena' is especially valuable to set-theoretic naturalists, insofar as it gives rise to and fully motivates the adoption of the meta-theoretic maxims 'unify' and 'maximize', which, in turn, justify the adoption of the axioms of set theory as our foundational theory, that is, a theory where all mathematical interactions among all mathematical objects take place.[18] Such a theory is ZFC (plus, possibly, LCs), interpreted as the theory of V.

Now, as a very general, overarching concern, one might legitimately doubt that the set-theoretic multiverse will fulfil the role of 'Generous Arena' equally well. The concern above may be summarised as follows:

Main Problem (Unification). *It is not clear whether and how the multiverse will fulfil the foundational role of a 'Generous Arena' (particularly, insofar as, at least*

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[16]Some such foundational roles are also re-stated by Maddy in the paper published in the present volume.

[17]Rather unsurprisingly, among the spurious ones, Maddy mentions 'Metaphysical Insight', that is, the pretension that set theory provides us with an account of what mathematical objects *really* are, and 'Epistemic Source', *viz.* the idea that set theory provides us with an account of what mathematical knowledge is.

[18]The maxim 'maximize' was also introduced by Maddy in (1996). The application of the maxim to our conception of V implies that this has as many 'objects' as possible. It should be noted that, while, theoretically, maxims may be in tension with each other, they ought to be seen as having the same foundational (normative) content (cf. Maddy 1997, pp. 211–2). Among other things, this is shown by the fact that the iterative concept of set (which is generally taken to motivate V) is also naturally construed as being 'maximal' (for this, also see Wang 1974, Boolos 1971 or Gödel 1947). Cf. also Maddy (1996), in particular, pp. 507–12.

prima facie*, the multiverse provides us with a disconnected picture of set-theoretic phenomena).*

This concern can even become more general. Along with 'Generous Arena', there are further foundational roles expressed by set theory when 'standardly' construed as the theory of V, that the multiverse may not be able to fulfil. Maddy describes her further concerns as follows:

> The choice between a universe approach and a multiverse approach is justified to the extent that it facilitates our set-theoretic goals. The universe advocate finds good reasons for his view in the many jobs that it does so well, at which point the challenge is turned back to the multiverse advocate: given that we could work with inner models and forcing extensions from within the simple confines of V, as described by our best universe theory, what mathematical motivation is there to move to a more complex multiverse theory? (Maddy 2017, p. 316)

The troubles expressed in the quote above may be re-phrased as follows:

Problem 1 (Foundational Roles) *While we know what the universe can do for us, we do not know what jobs the multiverse can do for us, in particular whether it can successfully carry out all and the same (foundational) jobs that the universe does.*

As is clear, the Main Problem as well as Problem 1 strike both 'instrumental' and 'ontological multiversism'.

There is, however, one issue which relates specifically to 'ontological multiversism', which Maddy sees as particularly worrisome: metaphysics is so much involved in the characterisation of this position (one need only consider Hamkins' statements that his own view is one of 'higher-order realism, that is, Platonism about universes'), as to make it highly unsuitable to the set-theoretic naturalist. Therefore, we have:

Problem 2 (Metaphysics) *Multiversism heavily relies on metaphysics, in a way that the set-theoretic naturalist does not view as legitimate.*

One further foundational role of set theory is 'Conceptual Elucidation', a role that set theory has often held in replacing muddled and unclear mathematical concepts with sharper ones. As examples of 'Elucidation', Maddy mentions the formulation of the concept of 'continuity' in the nineteenth century, and 'the replacement of the imprecise notion of function with the set-theoretic version [...]'.[19] Now, can this foundational role also be carried out by the multiverse? The 'ontological multiverse' practitioner thinks that one of the roles associated to the multiverse is precisely that of exploring different 'concepts of set', by examining the *structures* which instantiate them.[20] But, for the Maddian naturalist, this is a sharply different

[19]Maddy (2017), p. 293.

[20]See Hamkins (2012), p. 417. Hamkins says: 'Often the clearest way to refer to a set concept is to describe the universe of sets in which it is instantiated, and in this article I shall simply identify a set concept with the model of set theory to which it gives rise. By adopting a particular concept of set, we in effect adopt that universe as our current mathematical universe; we jump inside and explore the nature of set theory offered by that universe.'

way of construing ‘Conceptual Elucidation’, a way which is strongly tied to the metaphysics evoked in Problem 2, as concepts, in this case, are also taken to be *objective* entities (p. 312, 315–6).

The naturalist multiversist, then, has to face up with one further problem, which, like Problem 2, fundamentally relates to ‘ontological multiversism’:

Problem 3 (Concepts) *‘Conceptual Elucidation’, construed as elucidation of* set concepts *instantiated by universes in the multiverse, should not be seen as a legitimate foundational role of set theory.*

There is one last, and fundamental, problem, with the multiverse, which strikes both ‘instrumental’ and ‘ontological’ multiversism. We have already seen how different multiverse conceptions are motivated by different research programmes in set theory. Now, the mathematics is, maybe, ok, but it is not clear, to the set-theoretic naturalist, that the multiverse construct is really essential to develop it, and, if yes, whether it can really act as a fully formal theory of sets. Maddy says:

> I’m in no position to evaluate the mathematics: my question is whether multiverse thinking is playing more than a heuristic role, whether there’s anything that couldn’t be carried out in our single official theory of sets. If not, then it’s not clear that these examples give us good reason to incur the added burden of devising and adopting an official multiverse theory as our preferred foundational framework. (Maddy 2017, p. 316)

We may re-phrase the concerns above as follows:

Problem 4 (Axioms) *It is presently not clear whether the multiverse is just a useful* heuristic *tool, or whether it is really* instrumental *for pursuing our set-theoretic investigations and, in particular, whether it will be able to replace the currently standard* axioms *of set theory.*

3.4 Addressing the Problems

3.4.1 Phenomenology of the Multiverse

I start with addressing Problems 2 (metaphysics) and 3 (Concepts). As we have seen, a recurring metaphysical claim made by ontological multiversists is that universes in the multiverse *really* exist. This claim implies, among other things, that ‘extensions’ of V exist. As we shall see, this is an especially problematic claim to make.

One further, equally problematic metaphysical claim is Hamkins’ assertion that there are different ‘concepts of set’ (as well as different concepts of ‘ordinal’, ‘cardinal’, ‘power-set’, and so on), construed, once more, as platonic objects instantiated by other platonic objects (set-theoretic structures).

3.4.1.1 Platonism and Existence

Hamkins' conception seems to re-state what has been known, for some time, as Full-Blooded Platonism (FBP), that is, that particular kind of Platonism which implies the existence of a *plenitude* of mathematical (set-theoretic) objects, as many as posited by all conceivable theories T of sets.[21] Let us now try to assess whether and how FBP really impacts on the mathematics of the multiverse. For instance, let us take into account what Hamkins calls the 'ontology of forcing'.

We know what the main problem with forcing is: set-theorists often use the notation $V[G]$ to refer to 'forcing extensions' of the universe, but, as is clear, V already has *all* sets, so it is not clear how it could possibly be extended. Clearly, FBP would have us view $V[G]$ as a fully meaningful (and existent) object, provided we can define it in a consistent way.

Surprisingly, though, Hamkins' account of forcing does not use any consciously FBP-inspired metaphysical principle, but rather what he calls a 'naturalist' account of forcing, based on purely mathematical facts. A full mathematical analysis of this is beyond the scope of this paper, but some details may be provided.

Hamkins proves that:

Theorem 1 (Hamkins) *Given the universe of sets V, it is possible to define an elementary embedding $j : V \to \bar{V}$, where $\bar{V}$ is a definable class* in V*, and a $\bar{V}$-generic filter G, such that $\bar{V} \subseteq \bar{V}[G]$, and $\bar{V}[G]$ is also a definable class* in V.

The crux of the theorem is that $\bar{V}$ and $\bar{V}[G]$ are definable classes *in* V and, thus, the naturalist account of forcing consists in showing that one can code extensions of V with subclasses of V itself. Already at this stage, the issue of the existence of such objects as $\bar{V}[G]$ becomes, in a sense, irrelevant. It is true that, given Theorem 1, one may use FBP to re-inforce the idea that such objects as $\bar{V}[G]$ *really* exist, but it is clear that the metaphysical content of FBP, is not instrumental, per se, for the proof of theorem.

Thus, the set-theoretic naturalist may simply want to take note of the methodology invoked by the theorem, but entirely disregard the metaphysical content attributed to it by FBP. In Hamkins' own words:

> This method of application, therefore, implements in effect the content of the multiverse view. That is, whether or not the forcing extensions of V actually exist, we are able to behave via the naturalist account of forcing entirely as if they do. In any set-theoretic context, whatever the current set-theoretic background universe V, one may at any time use forcing to jump to a universe $V[G]$ having a V-generic filter G, [...]. (Hamkins 2012, p. 425)

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[21] FBP was introduced by Mark Balaguer in (1995). See also Balaguer (1998). Further details on Hamkins' use of FBP may also be found in Antos et al. (2015), pp. 2468–2470.

3.4.1.2 Concepts

There is a way to make sense of Hamkins'/the ontological multiversist's references to different concepts of 'set', 'ordinal', etc., as being instantiated by different set-theoretic structures, by bringing in a perspective wherein, ultimately, the metaphysical status of such concepts is irrelevant. In particular, I propose to construe the ontological multiversist's use of concepts within the framework of 'concept expansion'.

An especially promising and comprehensive account of concept expansion has recently been presented by Meir Buzaglo.[22] The main features of Buzaglo's account are: (1) concepts are flexible constructs; (2) the expansion of a concept is a law-like, forced process, that is, it is guided by the 'stretching' of some pre-established laws (axioms) which force the concept to evolve in a way which is *unavoidable* and, above all, (3) concept expansion gives rise to new objects.

Using such an account, Buzaglo shows that there are regulated ways to create new mathematical objects in an unavoidable way. At no point throughout this process, in Buzaglo's view, the mathematician has to assume that concepts are rigid, mind-independent *existents*: if they were such things, one could not make sense very easily of their 'extensibility'.[23]

Now, this account of concepts may be able to make sense of the ontological multiversist's invocation of different set concepts' being instantiated by alternative set-theoretic universes: the crux, here, is to see set concepts as mutually intertwined (that is, as arising from each other).

Again, the example of a forcing extension of the universe $V[G]$ will do. Using Buzaglo's account, the latter may just be seen as a 'new' set-theoretic object (and, correspondingly, set concept) arising from the regulated 'stretching' of the laws holding for the set concept of another object, V. After all, Buzaglo's account is not very far from Hamkins' own 'naturalist' account of forcing, whereby $V[G]$ is made fully *real* as a result of 'stretching' (laws holding for) V through forcing. Hamkins notes that this process may be compared to that relating to complex numbers, which are constructed by 'stretching' the square root function, in particular, by forcing it to have a value at $\sqrt{-1}$. Hamkins says:

> The case of forcing has some similarities [with that of complex numbers, *my note*]. Although there is no generic filter G inside V, there are various ways of simulating the forcing extension $V[G]$ inside V, using the forcing relation, or using the Boolean-valued structure $V^{\mathbb{B}}$, or by using the Naturalist account of forcing. None of these methods provides a full isomorphic copy of the forcing extension inside the ground model (as the complex numbers are simulated in the reals), and indeed they provably cannot–it is simply too much

[22]See Buzaglo (2002). Hints of a conception of concept expansion and evolution in mathematics may also be found in Lakatos (1976), which is also discussed by Buzaglo in the work mentioned.

[23]However, Buzaglo does not deny that realism about concepts may be compatible with concept change and evolution. See Buzaglo (2002), pp. 116–137, where the author examines Gödel's realist conception which, contrary to what has been stated above, takes concept evolution precisely as proof that concepts are objective constructs.

> to ask–but nevertheless some of the methods come maddeningly close to this. (Hamkins 2012, p. 420)

Now, this strategy may be extended to all other 'set concept-instantiating-universes' in the ontological multiverse, which may be construed as being new set-theoretic objects arising from stretching the laws (axioms) holding for concepts of other (previously established) set-theoretic objects.

3.4.1.3 Reality and Illusion

Leaving aside, for a moment, Hamkins' 'naturalist' interpretation of the multiverse, one could think that one of the purposes of the 'ontological multiverse' is that of making the illusion of 'living in separate, parallel *universes*' as fully real, which seems to involve some thorough-going metaphysical construal of set-theoretic practice.

However, in practice, what set-theorists working with the multiverse seem to be mostly interested in is something of a more definite mathematical character, that is, articulating a methodology which makes sense of 'accessing' other universes from a given universe. This, in particular, implies making sense of our experience of 'jumping', in Hamkins' own words, from one universe to another. This may be seen as a form of mathematical 'perspectivism', which may be described as follows:

Multiverse Perspectivism (MP) In working with one universe V, set-theorists are particularly interested in knowing: (1) what universes one may have access to from V and how, and (2) what V looks like from the point of view of those universes.

Now, as even multiversists may concede, one could certainly be a universist and implement MP within the single-universe conception, but MP is, trivially, facilitated by the adoption of the/a multiverse, as the multiverse precisely allows one to view (and refer to) universes as being 'separate', although somehow interrelated, constructs.

For instance, consider the following examples illustrating MP. The first two are multiverse axioms formulated by Hamkins:

Axiom 1 (Countability Principle) *Every universe V is countable* from the perspective of *a better universe W.*

Axiom 2 (Absorption into L) *Every universe V is a countable transitive model* in *another universe W satisfying $V = L$.*

One further example comes from the HP. As we have seen in Sect. 3.2.2, within the HP, it is essential that one can really refer to 'extensions of V'. In V-logic, one can prove the following fact:

Theorem 2 (S. Friedman) *If there is a small 'lengthening' of V, called $Hyp(V)$, then 'thickenings' of V may be viewed as outer models of V whose existence is* *implied by V-logic statements φ asserting: 'there is an outer model of V which satisfies T', where T is an extension of ZFC.*

So, it is only from the perspective of *Hyp(V)*, as defined in V-logic, that outer models of V can really be seen as existing.

In sum, the reference to 'reality' and 'illusion', thus, only serves to highlight more sharply the purposes and the extent of MP: the multiverse allows one to study inter-universe relationships by preserving our intuitive experience of this as a 'move' or a 'jump' from one universe to another.

3.4.2 Multiverse-Related Mathematics

I will now briefly present three case studies of 'multiverse-related' mathematics, which will help me illustrate that the multiverse may be able to fulfil many foundational jobs associated to set theory (which provides a response to Problem 1 (Foundational Jobs)), and also that it may not just be a useful heuristic, but rather a central construct in contemporary set theory, which partly responds to concerns expressed by Problem 4 (Axioms). I shall further address Problem 4 in Sect. 3.4.4.

3.4.2.1 Woodin's Set-Generic Multiverse and Ω-logic

I start with Woodin's results on CH in the set-generic multiverse. Woodin's guiding question was: is it possible to find an axiom which plays, for the structure $H(\omega_2)$, the same role as that played by PD for $H(\omega_1)$?[24] That is, is there any axiom which makes $H(\omega_2)$ 'well-behaved', as PD does with $H(\omega_1)$? Now, it turns out that it is relatively easy to force over properties of $H(\omega_2)$, which means that it is relatively easy to have different, mutually inconsistent pictures of $H(\omega_2)$. Therefore, Woodin identified the solution of the problem in identifying an axiom able to induce forcing-invariant properties of $H(\omega_2)$.

Now, it was known that many *forcing axioms* have this characteristic (that is, they are 'absoluteness axioms') and, moreover, that they imply the failure of CH. Therefore, Woodin's work was directed at identifying the appropriate forcing axiom which would make $H(\omega_2)$ well-behaved (and which, among other things, would also imply $\neg$CH), but the work carried out for this goal subsequently led to a parallel, equally fruitful, undertaking, that of defining a broader logical framework, wherein forcing invariance, in general, may be addressed. All this led Woodin to introduce a new logic, Ω-logic.

[24] $H(\kappa)$, for a cardinal κ, is the collection of all sets whose cardinality is hereditarily less than κ, that is, all sets whose elements and the elements of whose elements and so on, have cardinality less than κ.

Ω-logic is a logic in the full sense of the word, that is, a logical system which comes with its own definitions of semantic validity and logical consequence.[25] The semantics of Ω-logic is hinged on the use of Boolean-valued models $V^{\mathbb{B}}$ (where $\mathbb{B}$ is a complete Boolean algebra). The collection of all such models would, subsequently, become what Woodin defined the 'set-generic multiverse'.[26]

Now, it is important, for my purposes, to recall the definitions of validity and provability in Ω-logic. The definition of validity, with respect to a theory T, in Ω-logic (of $\models_\Omega$) is as follows:

Definition 1 (Ω-validity) $T \models_\Omega \phi$ if and only if, for all α ordinals, and all complete Boolean algebras $\mathbb{B}$, when $V^{\mathbb{B}}_\alpha \models T$, then $V^{\mathbb{B}}_\alpha \models \phi$.

The notion of provability is a lot more complex, as it uses *universally* Baire sets of reals, which cannot be addressed here.[27]

Definition 2 (Ω-provability) $T \vdash_\Omega \phi$ if and only if there exists an $A \subseteq \mathbb{R}$ universally Baire, such that $M \models \phi$, for every A-closed set M such that $M \models$ ZFC.

In turn, the Ω-conjecture is the conjecture that Ω-logic is complete (that is, that $\models_\Omega$ is equivalent to $\vdash_\Omega$).

As is known, in his (2001) Woodin, ultimately, focussed his attention on a specific forcing axiom, the $(\star)$ axiom, which allowed him to prove that, for all ϕ, ZFC+$(\star) \vdash_\Omega$ "$H(\omega_2) \models \phi$" or ZFC+$(\star) \vdash_\Omega$ "$H(\omega_2) \models \neg\phi$", precisely the kind of absoluteness result for $H(\omega_2)$ that Woodin was looking for.[28]

In the results mentioned, the foundational roles fulfilled by multiverse thinking are many. Woodin's 'Ω-logic solution' to CH looks very different from a 'standard', that is a solution consisting in showing that there is an axiom A which implies the truth or falsity of CH (something that Hamkins would subsequently define the 'dream solution' for CH).[29] The use of the Boolean-valued multiverse, or, more simply, of what would later become the set-generic multiverse, thus, stands out as an immensely successful way to elucidate statements of the complexity of CH, by

[25] Bagaria et al. (2006) is a comprehensive introduction to Ω-logic.

[26] Ω-logic makes its first appearance in Woodin (1999), and figures as a prominent tool in Woodin (2001). In those works, there is no direct reference to the set-generic multiverse, although the basic definitional ideas and concepts relating to it are already there.

[27] A rather accessible treatment of the provability relation in Ω-logic, and of universally Baire sets, is in Woodin (2011), p. 108.

[28] Moreover, Woodin was able to prove that:

Theorem 3 (Woodin) *ZFC*+$(\star) \vdash_\Omega$ "$H(\omega_2) \models \neg CH$".

It should be noted that the result requires the assumption of the existence of class-many Woodin cardinals, a particular strand of large cardinals having, as is known, far-reaching connections with Definable Determinacy Axioms, such as PD.

[29] See Hamkins (2012), p. 430.

showing, in particular, what such statements require in terms of ‘solving resources’, itself a way, in turn, to fulfil ‘Conceptual Elucidation’.

Secondly, multiverse thinking leads to define a broader logical environment, that of Ω-logic, through which statements like CH (or ¬CH), in particular, proof-theoretic and semantic facts about them, can be represented. This, among other things, also implies a re-structuring of the notion of proof in set theory, something which should be viewed as being strongly connected with two further foundational roles, ‘Shared Standard’ and ‘Metamathematical Corral’.

Finally, it should be noted that the kind of ‘multiverse logic’ inherent in the results mentioned is not only a specific ‘tool’ to be employed in representing facts about set-theoretic undecidability, but also a way to produce concrete mathematics, as shown by further work done on the Ω-conjecture.[30]

3.4.2.2 The Hyperuniverse Programme

As said in Sect. 3.2, the Hyperuniverse Programme (HP) has identified two main kinds of multiverse:

1. Zermelo’s height multiverse[31]
2. The hyperuniverse $\mathbb{H}$, that is, the collection of all countable transitive models of ZFC

How did the programme get there? First came the proof that certain maximality principles have very important first-order consequences. For instance, take the IMH:

Definition 3 (IMH) For all ϕ, if ϕ is true in an inner model of an outer model of V, then ϕ is true in an inner model of V.

On the one hand, the IMH implies that there are no large cardinals in V (only in inner models of V), and that PD is false. On the other, refinements of IMH, like SIMH#, imply, among other things, that CH is false.[32]

Now, HP’s maximality principles address extensions of V (in height and width). Therefore, one of the programme’s main goals from the beginning has been to clarify what mathematical resources are needed in order to express principles which address extensions of V, like the IMH.

The answer consisted in adopting the multiversist position that we have mentioned above. In particular, in order to make maximality principles *mathematically* expressible, HP turned to taking into account:

[30] See, for instance, Viale (2016).

[31] See Footnote 8.

[32] Further details on all the different maximality principles explored by the HP may be found in Friedman (2016).

1. A partially *potentialist* view of V, whereby height extensions are admissible, but the width of the universe is fixed, which accounts for the introduction of the multiverse (1), or
2. A fully *potentialist* view of V, that is, a view whereby V is extendible in height and width, which accounts for the introduction of the multiverse (2)

By adopting (1), one may only state the IMH syntactically, as, by (1), outer models of V aren't really available, whereas, if one adopts (2), in particular, if one takes V to be countable, then one may have *real* 'thickenings' and, thus, *real* outer models satisfying the IMH. The latter choice leads to the introduction of $\mathbb{H}$.

As is clear, then, multiverse thinking is fully integrated in the mathematics of the HP, in the sense that it would be a lot more cumbersome to express maximality principles such as the IMH within a universist framework.

Moreover, we could say that, also within the HP, the multiverse helps one fulfil tasks which are associated to 'Conceptual Elucidation', such as elucidate what V is like, and also foster one's mathematical investigations on maximality principles and their consequences. Therefore, as in the case of Woodin's set-generic multiverse, the introduction of the hyperuniverse is not merely a way to represent facts about 'truth in V': it is a way to produce new mathematics, which subsequently leads to finding solutions to outstanding set-theoretic problems.

3.4.2.3 The Multiverse Case for $V = L$

One further, striking example of multiverse-related mathematics is Hamkins' multiversist construal of $V = L$. A very influential and widespread view concerning $V = L$ is that the axiom would be a sort of *minimality principle*, that is, a principle which implies that V is as small as possible.[33] Hamkins has attempted to challenge this point of view, by using mathematical facts which are deeply connected with the multiverse.

First, let us contrast the following two conceptions:

Conception 1 *There is an* absolute *background concept of set, and of other set-theoretic notions, such as set, ordinal, cardinal.*

Conception 2 *There is* no *absolute background concept of set and of other set-theoretic notions, such as set, ordinal, cardinal.*

[33] Of course, the fact that L is 'minimal' is simply a mathematical fact (insofar as L is the smallest inner model of V). As is widely known, the view that construes L as a 'minimality principle' has been expressed by Gödel in (1947), p. 478–9. In that work, Gödel explicitly contrasts $V = L$ to *maximum principles*. Maddy herself, as is known, has argued in favour of the claim that $V = L$ would be 'restrictive'. The full argument may be found in Maddy (1997), pp. 216–232.

The 'ontological multiverse' view, as has been said many times, is bound up with Conception 2, which, in turn, suggests the following facts: L may not have the same ordinals as V, insofar as the concept '(ordinal)^L' may be different from the concept '(ordinal)^V'. By adopting this approach, one may, then, proceed to establish further striking mathematical facts, all of which suggest that $V = L$ is not inherently restrictive. What follows is a summary of Hamkins' argument:

1. By Shoenfield absoluteness, statements such as 'T has a transitive model', which are Σ^1_2, are *absolute* between V and L. Therefore, even theories T which contradict $V = L$ (for instance, the theory ZFC+'∃ a measurable cardinal') have transitive models in V if and only if they have transitive models in L.
2. All reals in V can be coded in a model M of ZFC+V=L.
3. Any transitive countable model M can be 'continued' such that it may, ultimately, become a model of ZFC+V=L. Thus, using *forcing*, one may first collapse any model to the countable, and then use such model to make it satisfy $V = L$.[34]

The upshot of this is remarkable: a case for the non-limitative character of V=L can be successfully made within multiverse-related mathematics in a way which is entirely in accordance with the set-theoretic naturalist's desiderata.

Again, while talk of concepts might look suspect to the naturalist, I have already construed their use in an essentially non-metaphysical way through 'concept expansion' (in Sect. 3.4.1.2 which addressed Problem 2), and, based on this perspective, we may elucidate such notions as that of 'constructibility' and also completely revolutionise our mathematical perspective on such axioms as $V = L$.

3.4.3 Opposing the Argument from Priority and a New Unification

The argument I will be reviewing in this subsection re-states concerns expressed by the Main Problem (Unification) and Problem 1 (Foundational Jobs), and, therefore, here I will mostly be concerned with these two problems.

In particular, one may see the Main Problem as being introduced by an argument, which I will call 'argument from priority', which states that the drive towards a *unifying* account of set theory was a primordial goal of set-theoretic research, already inherent in the process of axiomatisation carried out by such pioneers as Zermelo, Fraenkel, von Neumann and Skolem. By virtue of this fact, this goal should still be viewed as a major goal of set-theoretic research.

[34]This leads Hamkins to even surmise that: 'For all we know, our entire current universe, large cardinals and all, is a countable transitive model inside a much larger model of $V = L$.' (Hamkins 2012, p. 436).

The argument may more accurately be summarised as follows:

1. One main goal of the axiomatisation, since its emergence, was the drive towards producing a unifying account of set-theoretic phenomena[35]
2. The cumulative hierarchy V, which progressively emerged as the intended universe of discourse of the axioms, fully fulfilled this goal
3. Then came the 'multiverse phenomenon', triggered by the 'independence phenomenon'
4. The goal of unification, as expressed by 1., was viewed as *prioritary* over other goals in the early axiomatic set theory
5. Therefore, the universe view has *priority* over the multiverse view

There are, at least, two ways to resist the argument, both of which attack (4), and passing from (4) to (5). First of all, temporal priority is not adequate, as a criterion, to establish the absolute preferability of the single universe view over the multiverse view. In other terms, the fact that set-theorists *first* aimed at describing something like V and, *only later*, turned to considering a plethora of different structures does not imply that a single universe was (or is) something *inherently* more desirable.

A second way to dismantle the significance of temporal priority is provided by knowledge of further historical details. It is true that set theory was seen, already very early, as the 'theory of V', but the reasons which motivated this fact may be of a rather different kind than pre-supposed by the argument. One such reason was Zermelo's insistence on the determinacy of the concept of set, as expressed by the cumulative hierarchy and fully highlighted by his (quasi)-categoricity theorem. It should be noted that at the time when Zermelo presented the final version of the axioms there was no fully developed account or awareness of the potentialities of first-order logic, nor were axiomatic theories of other areas of mathematics (in particular, of arithmetic and analysis) naturally seen as being first-order theories.[36] The reason why the 'ontological problem' of set theory, so to speak, was seen by Zermelo as very pressing was the serious challenge to the determinacy of set-theoretic concepts posed by Skolem in the same years (something which had led to speculations about the 'relativity' of set theory). Therefore, (4), in the

[35]It is true, however, that this goal was not fully attained, arguably, until (Zermelo 1930). Of course, another prioritary goal of the axiomatisation was to prevent the formation of the paradoxes, something which Zermelo carried out through introducing the Axiom of Separation already in Zermelo (1908). For this and other aspects of the history of the axiomatisation of set theory, see Ferreirós (2010), in particular, pp. 297–324.

[36]At least, until first-order logic largely took over as a consequence, in particular, of Skolem's and Gödel's work. An exhaustive history of the triumph of first-order languages is provided by Shapiro in (1991), pp. 173–197. The relevance of 'categoricity arguments' for the 'triumph' of the single universe conception is also addressed in the same work, on pp. 250–259. A historically accurate reconstruction of the 'first-order proposal' for set theory is in Ferreirós (2010), pp. 357–64.

previous argument, should be assessed in the context of the fight between two early conceptions of set theory:

- Zermelian conception: there is one *determinate* concept of set, and one (associated) single structure instantiating it (this is the point of view of Zermelo in (1930))[37]
- Skolemian conception: there is no *unique* (and *fixed*) interpretation of set-theoretic concepts[38]

Now, while the dispute between these two viewpoints is, somehow, still alive, there is no need to see it as relevant to the Maddian naturalist's goals, and I'm rather inclined to think that adopting 'Generous Arena' based on Zermelo's concerns and conception would be, for such a naturalist, slightly embarrassing. Moreover, since, presumably, there is no need to follow Zermelo's strategy to successfully counter the argument for the relativity of set-theoretic concepts, one may also reject its main consequence, the support to the single-universe picture.

But then one could legitimately wonder what would be of 'unify' and unification concerns, if the latter ought not to be viewed as arising from the alleged determinacy of the concept of set and from the uniqueness of V. What follows aims to provide a tentative response to this.

There is a way to construe 'Generous Arena' along entirely different lines, which is suggested by Hamkins' foundational views. In a crucial quote from Hamkins (2012), Hamkins says:

> On the multiverse view, set theory remains a foundation for the classical mathematical enterprise. The difference is that when a mathematical issue is revealed to have a set-theoretic dependence, as in the independence results, then the multiverse response is a careful explanation that the mathematical fact of the matter depends on which concept of set is used, and this is almost always a very interesting situation, in which one may weigh the desirability of various set-theoretic hypotheses with their mathematical consequences. (p. 419)

Later on, he unpacks the methodology briefly sketched above, by addressing CH. He says that CH is a settled question, on the multiverse view. The reason would be that:

> The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extent to which it holds and fails in the multiverse, about how to achieve it or its negation in combination with other diverse set-theoretic properties. Of course, there are and will always remain questions about whether one can achieve CH or its negation with this or that hypothesis, but the point is that the most important and essential facts about CH are deeply understood, and these facts constitute the answer to the CH question. (p. 429)

[37] Of course this does not prevent one from interpreting Zermelo's conception of V in multiversist terms. See, again, Footnote 8.

[38] Although Skolemian relativism is a much stronger claim than this: it is the claim that set-theoretic concepts have no definite *theory-independent* meaning. Here, we cannot delve into the issue of whether Hamkins' FBP, ultimately, fosters such a form of anti-objectivism, as, for instance attributed by Field to Balaguer's FBP in Field (2001), pp. 334–5.

What Hamkins is setting out in the quote above is a fact already emerged in connection with Woodin's 'Ω-logic solution', that is, that finding a solution to CH implies taking into account the problem of what is needed, in terms of logical and mathematical resources, to solve it. However, this task cannot be executed, if one does not have a sufficiently broad collection of models (of a sufficiently strong theory) available, where the CH vs. $\neg$CH hypothesis may be tested, which is precisely what the multiverse provides us with.

Now, what I would like to highlight is the fact that the multiverse may, if viewed from such a foundational perspective, provide a different kind of unification, one which is needed to gain the sort of meta-theoretic 'knowledge' Hamkins is alluding to in the quotes above. In particular, one could suggest that the multiverse provides a different kind of 'Generous Arena' (a multiversist's 'Generous Arena'), one wherein all metamathematical interactions needed to provide us with knowledge about how to 'settle' problems of the same complexity, for instance, as CH, take place.

I argue that this may also be seen as a foundational role of set theory, which could be formulated as follows:

Multiversist's Generous Arena. *Set theory is also a systematic inquiry into the independence and unprovability phenomena, which provides us with knowledge about set-theoretic truth. In order to carry out such an inquiry, it is fundamental to have a unified metamathematical arena, where all relevant interactions take place*.

Of course, as already said, 'practically' one could still carry out such an inquiry within V. But should set-theorists agree that the Multiversist's Generous Arena is one further correct epistemological maxim for set theory to adopt, wouldn't the multiverse be the most natural candidate to fulfil it?

3.4.4 Relativism Reconsidered

Responses to the Main Problem (Unification), Problem 1 (Foundational Jobs) and, finally, Problem 4 (Axioms) are also provided by Steel's assessment of the goals of the multiverse in Steel (2014). We have already seen (in Sect. 3.2.2) that Steel's conception is an axiomatic version of the set-generic multiverse (MV).

Now, it is crucial, for Steel's purposes, to try to understand what MV really consists in. First, Steel shows that multiverse language is a sub-language of LST, that is, of the language of set theory. This is because a theorem proved by Woodin and Laver implies the following:

Theorem 4 (Woodin, Laver) *Given ϕ, $M^G \models \phi \leftrightarrow M \models t(\phi)$, where $t(\phi)$ is a formula saying: 'ϕ is true in some (all) multiverses obtained from M'.*[39]

[39] In Steel's notation, M^G is the set-generic multiverse containing all worlds satisfying MV.

The theorem says that there is a recursive translation from MV to LST given by $t(\phi)$, such that, given an MV-statement, there is always an LST-statement which also expresses it (whereas no inverse procedure to translate an LST-statement to an MV-statement is currently known). Now, there is a problem, however, with statements like CH. The reason is that CH, as is known, is not preserved by set-forcing in different worlds, therefore, CH cannot be expressed in MV.[40]

Although, on the one hand, this fact might be construed as showing that MV might lead to a loss of set-theoretic information, on the other hand, it might be seen as an indicator that the multiverse really is useful for us, insofar as it carries out an important task: that of drawing a line between ordinary set-theoretic statements which are 'meaningful', that is, are in the range of $t(\phi)$, and those which are not. But this is, again, a form of 'Conceptual Elucidation'!

But there's more. If Steel's MV really expresses the correct approach to set-theoretic truth, then one need abandon what Steel calls:

Strong Absolutism (SA) There is a reference universe, $\dot{V}$, which cannot be captured by MV.

As an alternative, one could take into account the following position:

Weak Relativism (WR) All propositions expressible in LST can also be expressed in multiverse language.

By WR, $\dot{V}$ makes sense if and only if it is expressible in multiverse language. However, WR still does not tell us whether one can really do that. The thesis that this is the case, a sort of combination of SA and WR, is called by Steel:

Weak Absolutism (WA) $\dot{V}$ makes sense, but as an *individual* definable world in the multiverse, which is *included* in all other worlds.

This latter standpoint looks particularly attractive to Steel, insofar as it introduces the view that the multiverse is reducible to one of its members, which he calls the *core*.

It should be noted that the acceptance of WA is, in turn, hinged on the acceptance of other mathematical conjectures. Steel has identified an axiom, the Axiom **H**, which singles out the core of the *multiverse* in a very detailed way.[41] With the addition of Axiom **H** to 'core truths', we might reach a situation wherein the

[40] However, it might still be the case that there are 'traces' of CH in MV, if one accepts additional hypotheses, such as the existence of a *core*, which is described later on, p. 22. See also Footnote 41.

[41] The Axiom says that V=HOD^M, where HOD is the class of all hereditarily definable sets, and M is a model of AD. Moreover, among other things, the axiom also implies CH. See Steel (2014), p. 171–177.

multiverse contains a preferred, ‘reference’ world. In practical terms, this is a way to revert to a partly universist account, which, in Steel’s view, provides a more ‘standard’ unification of set-theoretic phenomena, than the one provided by Hamkins’ multiverse addressed in Sect. 3.4.3.

Moreover, Problem 4 (Axioms) is also answered, in a sense, by Steel’s theory, insofar as MV is an axiomatic theory of the multiverse. It is true, though, that the theory, in itself, has never been used as a tool to make concrete set-theoretic mathematics. Furthermore, the status of the MV axioms (in particular, of some LCs) may be controversial. But this should rather encourage us to continue our investigations on the multiverse than retreat to the single-universe picture, as it is reasonable to predict that more developed axiomatic theories of the multiverse will respond more fully to the concerns expressed by Problem 4.

3.5 Concluding Remarks

Let’s take stock. My goal has been that of trying to explain away the Maddian naturalist’s concerns about the set-theoretic multiverse by, essentially, adopting the following two points of view: (1) metaphysics, even when explicitly evoked by multiverse theorists, is not fundamental for a successful articulation of the multiverse and (2) multiverse practice has led to important mathematical discoveries, which have a robust foundational impact and, moreover, show that a multiverse theory may be able to fulfil the many foundational roles associated by the set-theoretic naturalist to set theory, including that of ‘unification’.

This work has been concerned to a large extent with multiverse practice, without providing a sharp definition of it. It should be noted that ‘multiverse practice’ does not just consist in dealing with models of set theory, but rather implies manipulating mathematically a determinate multiverse framework from the beginning, with a view to pursuing a broad, but clearly set out, range of mathematical goals, such as the study of: (1) inter-universe relationships, (2) axioms for the multiverse itself, (3) principles which are formulated in a multiversist language, and (4) problems whose complexity requires a strictly multiversist construal.

In Sect. 3.2 we have seen that the multiverse may not be seen as a single strand of mathematical practices and methodologies. Furthermore, we do not currently know whether there will ever be an ultimate conception of the multiverse. In some respects, ‘ontological multiversism’ may be seen as such a conception, insofar as: (1) it is maximally broad, and (2) it is particularly flexible, as it allows for the highest amount of ‘perspectivism’. However, ‘ontological multiversism’ is also the most controversial and problematic conception among those examined, for the reasons we have reviewed in Sect. 3.3.

From time to time, it has been made clear that there may be no specific task that the universist may not try to successfully emulate within their single V. However, it is, in my view, rather apparent that the multiverse enormously facilitates fundamental practical tasks, so why would the conscientious set-theoretic naturalist, ultimately, oppose this fact?

There is surely one main concern which has not been fully dispelled by the paper, that relating to the absence of an axiomatic theory of the multiverse. However, I have suggested that further mathematical research may eventually allow us to fully axiomatise the multiverse, along the lines of what has been attempted by Steel with MV. As is clear, this would help the Maddian naturalist to defeat more convincingly the concerns expressed by Problem 4.

However, even now, and notwithstanding the current absence of an axiomatic framework, I have suggested that the multiverse may already provide us with the same foundational benefits that the single-universe picture does, something which should help fully dissipate the set-theoretic naturalist's concerns about its usefulness.

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Reply to Ternullo on the Multiverse

Penelope Maddy

Doctor Ternullo raises a host of important issues, but I focus here on the central theme: his defense of the multiverse from the point of view of a naturalist, indeed, a naturalist of a particular variety that I call a 'second philosopher'. In a recent paper on the foundations of mathematics (a companion piece to my contribution to the present volume), I considered the possibility that some sort of multiverse theory could replace our current set theory in a range of foundational jobs now performed by ZFC + Large Cardinals (LCs). I concluded that for now, in the current state of knowledge, it isn't clear that this move is feasible or advisable. Ternullo apparently disputes this conclusion: 'the multiverse may be as acceptable as the universe for … the foundational purposes of set theory' (Ternullo 2019, p. 46).

In an odd twist, though, Ternullo argues that one fundamental aspect of the foundational goal, the job I call Generous Arena,[1] is itself misguided, that the argument for it is, for my naturalist, 'slightly embarrassing' (Ternullo 2019, p. 63). As it happens, leading multiverse theorists don't see the situation this way; they embrace this foundational goal and argue that their theories meet it. For example, this from Hamkins:

> The multiverse view does not abandon the goal of using set theory as an epistemological and ontological foundation for mathematics, for we expect to find all our familiar mathematical objects … inside any one of the universes of the multiverse. (Hamkins 2012, p. 419)

[1] Ternullo also uses the term 'Generous Arena' for a different idea (Ternullo 2019, pp. 63–64), but in the passage under discussion here, he's concerned with the sense delineated in my contribution to this volume.

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And here's Steel:

> We want one framework theory [i.e., foundational theory], to be used by all, so that we can use each other's work. It is better for all our flowers to bloom in the same garden. If truly distinct frameworks emerged, the first order of business would be to unify them. … The goal of our framework theory is to *maximize interpretive power*, to provide a language and theory in which all mathematics, of today and of the future so far as we can anticipate it, can be developed. (Steel 2014, pp. 164–165)

Ternullo seems to reject this approach, on the grounds that the argument for the foundational goal itself is flawed (Ternullo 2019, pp. 61–63).

That argument, as he sees it, rests on the claim that Generous Arena was present early on in the history of set theory – as it was – and that this temporal priority implies logical priority. I think we can all agree that this is a weak argument, but it's not the right argument, as I hope is clear from §I of my contribution here.[2] Regardless of when they first arose, the mathematical attractions of Generous Arena, along with Risk Assessment, Metamathematical Corral, and Shared Standard, remain as strong today as ever. In his willingness to forgo Generous Arena, Ternullo sounds a theme familiar, as we've seen, in category-theoretic and univalent foundations, but not one we hear from multiverse theorists like Hamkins and Steel. This creates some mismatch in the debate between Ternullo and me, since my analysis is explicitly addressed to the question of the multiverse's aptitude for filling this foundational role (among others) and he seems to think it needn't be filled. Still, there are some surrounding points well worth considering.

I begin, as Ternullo does, by attending to various versions of multiversism, though now with an eye not to the mathematical differences between them, but to the different philosophical or methodological stances one might take toward them. §II sketches my concerns about the multiverse in a foundational role, and §III considers other significant roles that multiverse thinking might play.

I. What Is a Multiverse View?

Ternullo begins with a straightforward characterization of the central distinction:

> Whether set theory should be interpreted as the theory of a *single* universe of sets … or … as a theory about *multiple* structures … that is, about a set theoretic 'multiverse'. (Ternullo 2019, p. 43)

He alludes to Koellner's distinction between pluralism and non-pluralism:

> *pluralism* … maintains that … although there are *practical* reasons that one might give in favor of one set of axioms over another – say, that it is more useful for a given task –, there are no *theoretical* reasons that can be given …
>
> *non-pluralism* … maintains that the independence results merely indicate the paucity of our standard resources for justifying mathematical statements. … theoretical reasons *can* be given for new axioms. (Koellner 2014, p. 1)

[2]See §I of [2017] for a bit more detail.

Multiversism, then, is a model-theoretic version of pluralism:

> There is not a single *universe* of set theory but rather a *multiverse* of legitimate candidates, some of which may be preferable to others for certain practical purposes, but none of which can be said to be the 'true' universe. (Koellner 2013, p. 3)

Ternullo agrees that multiversism is 'an *ontological* (*semantic*) variant' of pluralism (Ternullo 2019, p. 44). Presumably he also takes his multiversist to hold that, though there might be practical reasons for preferring one truth value for an indeterminate statement over another or one universe of the multiverse over another, there are no theoretical reasons for this – there is no determinate truth value, no 'true' universe.

Now Ternullo notes that unabashedly metaphysical views like these may be problematic for the naturalist, but he concludes that the metaphysics can be disregarded (Ternullo 2019, §4.1). He has his own reasons for saying this, but in any case, we agree on the underlying point, that metaphysics is largely irrelevant to the mathematical issues at hand. In the hope of clarifying some of these matters, let me sketch a rough taxonomy of philosophical stances on multiversism, beginning from the most ontologically or semantically loaded and moving on from there.

On the deeply metaphysical end of the spectrum, there's Hamkins's position:

> With forcing, we seem to have discovered the existence of other mathematical universes, outside our own universe, and the multiverse view asserts that yes, indeed, this is the case. (Hamkins 2012, p. 425) Each … universe exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist. (Ibid., pp. 416–417)

In Koellner's terms, presumably this Metaphysical Multiversism[3] takes 'theoretical' considerations to tell us something about the structure of its generous ontology, perhaps, for example, that every world thinks ZFC.[4] It might seem that the theoretical/practical distinction coincides with the intrinsic/extrinsic distinction familiar in the philosophical foundations of set theory – where intrinsic considerations are somehow intuitive, or self-evident, or contained in the concept of set, and extrinsic considerations involve attractive consequences or interrelations or something of that sort – but I think this can't be right. Koellner writes that …

> given the current state of our knowledge a case can be made for being a non – pluralist about ZFC and large cardinal axioms (Koellner 2013, p. 4)

[3]I use this term in place of Ternullo's 'ontological multiversism' to leave room for a position that replaces objective entities with determinate truth values.

[4]Steel (2014) and Woodin (2011) both take ZFC to be true in every world of the multiverse, but Hamkins sometimes does not: 'There seems to be no reason to restrict inclusion only to ZFC models, as we can include models of weaker theories ZF, ZF^-, KP, and so on, perhaps even down to second-order number theory' (Hamkins 2012, p. 436). On the other hand, we've seen that he addresses the foundational goal like this: 'we expect to find all our familiar mathematical objects … inside any one of the universes of the multiverse' (ibid., p. 419), which would seem to require at least ZFC (and if Risk Assessment is taken into account, large cardinals would be handy as well). In any case, Hamkins certainly embraces a number of objective truths ('multiverse axioms') about the multiverse in §9 of his [2012].

. . . in other words, in multiverse terms, for assuming these axioms across all worlds of the multiverse, and from his other writings, it's clear that some of this case is extrinsic. Similarly, Hamkins allows that

> the mathematician's measure of a philosophical position may be the value of the mathematics to which it leads (Hamkins 2012, p. 440).

So it appears that at least some extrinsic considerations must also yield information about the multiverse, must also be included under 'theoretical'.

Suppose, then, that a certain set-theoretic statement, perhaps a candidate for a new axiom to be true in all worlds, or perhaps another sort of general claim about the structure of the multiverse, has many mathematical advantages and no mathematical disadvantages. This wouldn't be enough for the Metaphysical Multiversist to endorse it, because we'd have to be confident that those mathematical merits produce theoretical support, not mere practical support. We'd have to be confident that our belief in that set-theoretic statement, however attractive it might be, isn't just wishful thinking, that the objective mathematical realm of the multiverse doesn't just happen to deny us something we'd very much like to have. Many philosophers, in the tradition of Benacerraf's famous challenge to Platonism, would ask how we could come to know that our beliefs are tracking the truth about an abstract realm. My naturalist asks a question that's logically prior to Benacerraf's: why should we demand more than mathematical merits? Why should those mathematical merits be held hostage to extra-mathematical metaphysics? Her answer is that this is wrong-headed, that the compelling mathematical reasons should be enough all by themselves.[5]

Though Metaphysical Multiversism is uncongenial to the naturalist, as Ternullo says, this isn't the end of the story; there are other varieties of multiversism. We could, for example, leave metaphysics aside and simply talk about theories. Set theory, then, isn't the project of describing an abstract mathematical realm; it's the project of forging a powerful mathematical theory to serve the foundation goal (among others).[6] The universist advocates ZFC and its extensions in this role; the multiversist proposes an alternative multiverse theory of sets and worlds to take its place. For these purposes, all extrinsic considerations would be on equal footing; there'd be no distinction between 'theoretical' or truth-tracking cases and 'merely practical' cases. This Theory Multiversism is an option entirely open to the naturalist, should the evidence point that way.

A final variant sees the multiverse as analogous, not to a universe ontology, not to a universe theory, but to the iterative conception. In universe thinking, the iterative conception serves as an intuitive picture that helps us see our way around in deriving consequences from the axioms or seeking new avenues for axiom choice. From

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[5]See [2011], pp. 55–59. In that book, I propose an alternative metaphysical position, Thin Realism, that avoids this problem by essentially reading its ontology off the analysis of proper methods, including extrinsic methods, but I doubt this is what Hamkins or Koellner or Ternullo has in mind.

[6]We could think of this as the project of forming an optimally effective concept of set. Cf. the Arealism of [2011].

the naturalist's non-metaphysical perspective, intrinsic considerations based on this picture are potentially of great heuristic value; the history of the subject amply demonstrates what an immensely successful tool the iterative conception has been. But, for the naturalist, it's important to stress that the value of this intuitive picture rests on the great mathematical merits of the work it's inspired, in other words, on its extrinsic success.[7] If we were presented with an alternative intuitive picture that conceptualizes set-theory differently, if that alternative way of guiding the subject were more fruitful than the iterative conception, we should switch our allegiance without regret.[8] The Heuristic Multiversist[9] argues that the intuitive picture of a multiverse is just such an alternative; he might propose that ZFC and its extensions, guided by the iterative conception, should be replaced with a multiverse theory based on the new picture. This would be a version of Theory Multiversism, but other possibilities emerge in §III below. Either way, the basic suggestion is that the intuitive multiverse picture would guide the practice in new and different directions with important mathematical advantages.

There are no doubt other ways to frame a philosophical perspective for multiversism, and perhaps predictably, one prominent multiverse theory, the one due to Steel, doesn't fit squarely in any of the three bins just described. As Ternullo notes, Steel is out to explore whether CH is 'meaningful'; his multiverse theory is intended, not as an alternative subject matter (Metaphysical Multiversism), not exactly as an alternative theory (Theory Multiversism), but as a way of determining which, if any, sentences in the language of set theory (not the multiverse language of sets and worlds) are 'meaningless', pose 'pseudo-questions'. How he goes about this and what conclusions he draws are quite subtle matters that go well beyond the scope of this reply.[10] Still, I hope these three rough categories will help illuminate the debate between Ternullo and me. As this is an intramural debate between naturalists, we're focused primarily on multiversisms of the Theory and Heuristic varieties.

II. Naturalistic Concerns About Multiversism

In the paper Ternullo is discussing, I raise a number of questions about multiverse theories as potential alternatives to ZFC and its extensions as our basic foundational theory. The most fundamental of these is that a foundational theory, as we now understand it, has to be a theory, has to be an explicit set of axioms capable of

[7]See [2011], pp. 131–137.

[8]Something like this actually happened when the intuitive picture of sets as extensions of properties fell out of favor in light of its conflict with the extremely fruitful axiom of choice.

[9]Ternullo contrasts 'heuristic' with 'instrumental' (2019, p. 53). See §III below.

[10]Toby Meadows and I hope to clarify some of these matters in 'A philosophical reconstruction of Steel's multiverse', in preparation. I also neglect the hyperuniverse program, simply because I don't understand it well enough to comment.

doing the foundational jobs. Of the multiverse accounts on offer, only Steel's comes with a set of axioms, a fully explicit first-order theory of sets and worlds, but as noted, his goal is to evaluate the sentences of ordinary set theory, not to replace them with something different. So, flat-footed as it sounds, the general lack of an explicit multiverse theory strikes me as a serious obstacle to a new and different multiverse foundation.

Hamkins's stand on the foundational status of the multiverse was quoted above:

> We expect to find all our familiar mathematical objects ... inside any one of the universes of the multiverse. (Hamkins 2012, p. 419)

Roughly speaking, it seems any world of the multiverse can serve as our Generous Arena, and ZFC (satisfied by that world) as our Shared Standard. Presumably Risk Assessment is to be carried out in a world with large cardinals, that is, in ZFC+LCs. There's some question about Meta-mathematical Corral: if we only care about corralling a generous arena, we're once again thrown back on ZFC and its extensions; if we want to corral all of mathematics, it seems we'd need a theory of our multiverse, which we've seen Hamkins's doesn't provide. On Steel's view, ZFC+LCs turns up in the meaningful part of set-theoretic language and continues to carry out its usual foundational functions. For the most part, then, ZFC and its extensions retain their foundational roles – in that respect, no alternative is actually on offer. So it's hard to see a case for replacing a universe view with multiverse view for foundational purposes.

But this isn't the end of the story. Some version of the multiverse perspective may have such attractive mathematical features that we're moved to adopt it even if a familiar theory like ZFC remains our official foundation. Ternullo mounts a case along these lines.

III. Ternullo's Defense

One striking turn in Ternullo's discussion is his characterization of Zermelo's famous 'On boundary numbers and domains of sets' (Zermelo 1930) as 'the first description of a multiverse' (Ternullo 2019, p. 47).[11] If this were so, it would go a long way toward showing that multiversism has important and far-reaching mathematical consequences! Working in a strong implicit meta-theory, Zermelo presents an analysis of 'normal domains' characterized by second-order ZFC minus Infinity[12]: their 'boundary numbers' are inaccessible cardinals; they can be

[11] To be clear, Ternullo isn't claiming that the historical Zermelo understood his work in multiverse terms (see Ternullo 2019, p. 47, footnote 8). He holds, rather, that Zermelo is a 'height potentialist' and that this position can be seen as a kind of 'height multiversism' (see Footnote 13 below).

[12] He leaves out the axiom of infinity to allow for a 'finitary' normal domain acceptable to intuitionists (so for him ω is a boundary number). As he sees it, a generous store of normal domains makes set theory adaptable for a wide range of applications.

decomposed into ranks up to that number (this is touted as one of the extrinsic benefits of Foundation); any two with the same boundary number are isomorphic; for any two with different boundary numbers, one is an initial segment of the other. The question then arises: are there any normal domains, are there any boundary numbers? Zermelo mounts an argument in the meta-theory that for any ordinal α, there's a corresponding boundary number κ_α; in modern terms, he's argued for the Axiom of Inaccessibles:

> We must postulate the *existence of an unlimited sequence of boundary numbers* as a new axiom for the 'meta-theory of sets'. (Zermelo 1930, p. 429)

Though Zermelo does take second-order 'ZFC-Infinity' to characterize each of an unending series of normal domains, I see no evidence that he intends his second-order 'ZFC + a proper class of inaccessibles' in the meta-theory as anything other than a description of the single universe in which all these normal domains reside.[13] If including an axiom of inaccessibles is enough to qualify a list of axioms as a multiverse theory, then almost all set theorists these days are multiversists; this sets the bar far too low, renders the term useless. So it seems that Zermelo is best left out of this discussion.

Ternullo is on stronger ground when he extends the appeal to mathematical consequences into contemporary set theory. I'm happy to grant that, for example, Hamkins's multiverse thinking has led to a fruitful investigation of 'set-theoretic geology' or that Steel's approach has focused attention on important questions about the 'core'. Cases like these display a clear heuristic benefit to thinking in terms of an intuitive multiverse picture – on this Ternullo and I agree – but he goes on to insist that these benefits aren't purely heuristic, that they are actually 'instrumental'. He draws this distinction from a question raised in my paper: can all the welcome mathematics inspired by multiverse thinking be carried out in our familiar universe theory, that is, are these all theorems of ZFC and its extensions? If the answer to this question is no, then multiverse thinking would be more than merely heuristic – fully instrumental, in Ternullo's terms – but as far as I can tell, the answer is yes, which Ternullo seems to acknowledge:

> It has been made clear that there may be no specific task that the universist may not try to successfully emulate within their single V. However, it is, in my view, rather apparent that the multiverse enormously facilitates fundamental practical tasks. (Ternullo 2019, p. 66)

This is just to say that multiverse thinking is of great (purely) heuristic value.

Notice that we have here instances of Heuristic Multiversism different from what was envisioned in §I: the multiverse picture isn't being used to inspire new axioms toward a version of Theory Multiversism, but to inspire new mathematics, new

[13] Many observers see Zermelo as a potentialist, but I have my doubts. Though everyone, actualist and potentialist alike, uses a familiar range of metaphors – the universe is unending, etc. – it seems to me that the cash value of potentialism is the rejection of quantification over all sets. But this is exactly what Zermelo seems to do, for example, in arguing that there's an inaccessible for every ordinal.

concepts and methods, within our existing theory of ZFC and its extensions. And there's another potential contribution of Heuristic Multiversism, as well. Hamkins observes that

> There is no reason to consider all universes in the multiverse equally, and we may simply be more interested in parts of the multiverse consisting of universes satisfying very strong theories, such as ZFC plus large cardinals. (Hamkins 2012, p. 436)

Now the process of narrowing down to a restricted range of worlds may well be functionally equivalent to the process of adding new axioms to ZFC, so what's of interest here is the suggestion that thinking in multiverse terms could bring new and different considerations to bear on that process. In other words, multiverse thinking might help us to refine our official theory of sets. In fact, this may be the ultimate upshot of Steel's line of thought: a stretch of multiverse thinking leads him to propose a new axiom for ordinary set theory.

In sum, then, our naturalist has no straightforward form of Theory Multiversism, only Steel's set of axioms with a different motivation, but there seems to be ample room for Heuristic Multiversism to do significant mathematical work in a number of different ways. We can draw two morals. The first is that ZFC and its extensions aren't uniquely tied to the intuitive universe picture of the iterative conception. They could be thought of, instead, as the shared theory of a range of worlds in the multiverse, so that what the universist sees as adding new axioms about V, could instead be seen as a narrowing of the range of worlds we take to be of interest. The second moral is one that should appeal to Ternullo's naturalism: since these intuitive pictures, universist and multiversist, are playing a merely heuristic role, there's no reason at all not to exploit them both, no reason at all not to switch back and forth depending on which is more suggestive in a given context. In the end, set theorists should feel entirely free to think in *any* intuitive terms that can lead them to good mathematics!

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