Peano's Conception of a Single Infinite Cardinality

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Abstract

While Peano's negative attitude towards infinitesimals, in particular, *geometrical* infinitesimals, is widely documented, his conception of a *single infinite cardinality* and, more generally, his views on the *infinite*, are a lot less known. The paper reconstructs the evolution of Peano's ideas on these questions, and formulates several hypotheses about their underlying motivations.

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1 Introduction

Peano is widely known for his contribution to the creation of modern logical symbolism and to the axiomatisation of arithmetic, in particular, for the axioms which still bear his name.[1] His proof of the impossibility of *infinitesimal segments* in geometry has also received the due attention;[2] a lot less, if at all, explored are, on the contrary, Peano's views on the *infinite*, an area which, precisely when Peano was most active mathematically, had experienced major and

[1]For an overview of Peano's contributions to logic, see (Kennedy, 1980), (Borga et al., 1985), (Grattan-Guinness, 2000), Ch. 5, and the 2021 special issue of *Philosophia Scientiae* 'Peano and His School' edited by P. Cantù and E. Luciano.

[2]Cf. (Peano, 1892a). For a reconstruction of Peano's views about the infinitesimals, see (Ehrlich, 2006), and the more recent (Freguglia, 2021).

unprecedented breakthroughs, especially thanks to the work of, just to mention a few names, Cantor, Dedekind, Schröder, Veronese and other mathematicians.[3] Recently, Peano's early conception of a *single infinite cardinality* has been scrutinised by (Mancosu, 2016), which, besides showing where the conception stands in the history of the infinite, also casts 'Peano's Principle' as a *bona fide* cardinality principle able to put pressure on the neo-logicist doctrine of the *analyticity* of Hume's Principle.[4]

(Mancosu, 2016) also lists the different stages of the evolution of Peano's conception, and demonstrates that Peano's ideas became increasingly closer to,

[3]Cf., again, (Grattan-Guinness, 2000) and (Ferreirós, 2010).

[4] Hume's Principle [HP], taken by (Frege, 1884) to be the correct statement of 'numerical identity' and, as a consequence, of the concept of number, is the universal closure of the following (second-order) principle:

$$\#x : F(x) = \#x : G(x) \leftrightarrow F \approx G$$

'the number of Fs is equal to the number of Gs if and only if F is equinumerous with G (i.e., if Fs and Gs can be put in a one-to-one correspondence)'. Mancosu has called the formalisation, in second-order logic, of Peano's conception of a single infinite cardinality 'Peano's Principle':

$$\#x : F(x) = \#x : G(x) \leftrightarrow (\neg Fin(F) \wedge \neg Fin(G)) \vee ((Fin(F) \wedge Fin(G)) \wedge F \approx G)$$

'the number of Fs is equal to the number of Gs if and only if either F and G are infinite or, if finite, equinumerous', and discusses it in connection with the 'good company problem' for HP, see (Mancosu, 2016), Ch. 4.

and from a certain point onwards, indistinguishable from, Cantor's, but it does not pin down the exact motivations behind the conception itself, nor does it explain why Peano, ultimately, endorsed Cantor's *transfinite*.[5]

The purpose of this paper is to fill in the gap, by providing, in a way as far as possible based on the extant textual sources, a more articulated account of the evolution of Peano's ideas, which might also shed light on the potential significance of Peano's early conception.

With respect to the latter point, our interpretation of Peano's doctrine will seek to establish connections between Peano's and Galileo's conceptions. Galileo's argument in (Galileo, 1638) was that the tension between the demands of Euclid's Axiom (the Part-Whole Principle) and those of the 'bijection method' which Galileo himself discusses, and uses in a positive way, maybe for the first time in the history of mathematics, to establish the *equinumerosity* of the *natural* and *square* numbers, cannot be overcome.[6] As a consequence, Galileo concludes that one should surrender to the idea that there can be no *measure* of actually infinite collections, as there is of finite ones.

That Peano had quandaries comparable to Galileo's may be deduced both from his lukewarm adherence to (and, probably, incomplete understanding of) the Cantorian theory, as well as from his reliance on other kinds of infinitary intuitions, of a *geometrical* character, such as those involved in the construction of the Peano curve. Even when his understanding of the Cantorian theory is

[5]Cf. (Mancosu, 2016), pp. 165-6, in particular fn. 21.

[6]Both Euclid's Axiom and the bijection method are discussed in depth in section 4.1.

adequate, he seems to want to use the theory in a way which suits best his own purposes and 'Galilean' conception.

The structure of the paper is as follows. In the next section, we make some preliminary considerations, in §3 we examine the evolution of Peano's ideas and, finally, in §4, we consider three possible motivations behind Peano's conception.

2 Did Peano Have A Definite Conception of the Infinite?

From reading Peano's terse statements on the subject, one wonders whether the Italian mathematician had *any* determinate conception of the infinite at all; sometimes, it would rather seem that Peano's ideas are just insufficiently developed to be seen as advocating any specific view.

On some authors' judgement, Peano, while brilliant at spotting inaccuracies, and in improving on already established results, was poorly armed to devise (or just propose) a new theory, and rarely took enough care of exposing in full detail, let alone justifying, conceptions he happened to champion.[7] However, there is (some) evidence that this might not be the case with reference to the question of the infinite.

To begin with, Peano actively engaged in the lively debates on the nascent theory

[7] See, for instance, (Grattan-Guinness, 2000), p. 221, and Agazzi's introduction to (Borga et al., 1985), p. 7. For a different assessment of the significance of Peano's work, see (Rodriguez-Consuegra, 1991), especially the beginning of Ch. 3.

of sets, and on infinitesimals, and relentlessly discussed aspects of these topics with Cantor, Russell, Frege, Veronese and other mathematicians.[8] Moreover, as already mentioned, Peano forcefully opposed Veronese's (geometrical) infinitesimals, by producing a (purported) proof of their inconsistency,[9] a fact which, considered on its own, already demonstrates that he was, at least, willing to go to great lengths to understand questions about the nature of the infinite, both philosophically and mathematically.

Peano's ideas gradually became more and more akin to Cantor's, to the point that no traces of his earlier conception can be found in later work. What motivated such a noticeable change of mind? We believe that Peano became convinced of the intrinsic weakness of the alternatives to Cantor's conception, including his own, precisely as a consequence of his engagement in the foundational debates which were hosted by, among other journals, his *Rivista di Matematica*. The soundness of this interpretation is further corroborated by the content of Peano's correspondence with Cantor in the year 1895, which, in our view, proved instrumental for Peano's 'conversion' to the Cantorian approach as much as, eventually, his collaboration with the arguably most set-theoretically-minded member of his school, Giulio Vivanti.

[8]Traces of such discussions may be found in, among other works, the many letters that Peano exchanged with his mathematical interlocutors, including Cantor (see later, section 3.2). Almost all of Cantor's letters to Peano may be found in (Cantor, 1990). The correspondence between Peano and other mathematicians is reproduced in (Peano, 2008).

[9]See fn. 2 and section 4.3 of the present paper.

However, even after inspecting much primary and secondary literature, as we have done in this paper, the full motivations behind Peano's early conception and the shift to the Cantorian conception, still appear, to some extent, mysterious, but the interpretative hypotheses we shall formulate in §4, will provide us with, at least, some clues.

3 The Evolution of Peano's Conception

In this section we follow closely the evolution of Peano's ideas between 1891 and the turn of the century. For this, we avail ourselves of three fundamental textual sources:

1. Peano's own articles, in particular, (Peano, 1891), where his early conception of the infinite is first formulated, and (Peano, 1892a);

2. Peano's 1895 correspondence with Cantor, consisting of six letters from Cantor to Peano;[10]

3. the first three volumes (and sections thereof) of his *Formulaire de mathématiques*, published in rapid succession between 1895 and 1901 ((Peano, 1895), then (Peano, 1897), (Peano, 1898), (Peano, 1899), (Peano, 1901)), where both Peano's early and 'Cantorian' conception may be found.

[10]Peano's letters to Cantor are no longer available, as confirmed, in a private email to the authors (23/02/2022), by Clara Silvia Roero, who has recently re-edited Peano's works and correspondence (cf., again, (Peano, 2008)).

We close this section by briefly discussing Vivanti's role in the evolution of Peano's ideas.

3.1 'On the Concept of Number'

In 1891, Peano published his famous article 'Sul concetto di numero' ('On the Concept of Number') in the *Rivista di matematica*, the journal he had founded the same year.

Two years after *Arithmetices principia, nova methodo exposita* (Peano, 1889), in which Peano first presented his axioms of the natural numbers, in this work, among other things, he simplifies his notation, demonstrates that his famous five postulates of the natural numbers are mutually independent, and builds a more general axiomatic system of numbers by also taking into account *relative*, *rational* and *real* numbers.[11]

(Peano, 1891) is Peano's first endeavour to deal explicitly with the infinite. In §9, he defines a function, '$num\ a$', whose domain consists of 'classes' (denoted $a, b, c, ..., u, ...$), and whose values are the 'cardinalities' of these classes; in Peano's own words, $num\ a$ is '*the number of elements of the class* a'[12].

Now, if a is a finite class, then $num\ a$ is just the (finite) number of its elements, i.e., a *natural number* n. But then, Peano states that $num\ a$ is not always a

[11] Cf. (Borga et al., 1985), pp. 79-94, and (Rodriguez-Consuegra, 1991), Ch. 3.2.

[12] [Con *num a* intenderemo "il numero degli individui della classe *a*" (p. 100).] Page numbers of (Peano, 1891) are those of (Peano, 1959). The English translations of Peano's quotes are all ours.

natural number, since the set of natural numbers does not include 'zero' and 'infinity'.[13] This is the first time Peano mentions the possibility that a class a be *empty*, or *infinite* (that is, that $num\ a = \infty$).[14]

We wish to say something more substantial about '∞'. Peano seems to take it to be a *bona fide* 'infinite quantity', which can be manipulated like any other (finite) quantity, as is clear from the propositions 3 and 4 in §9:

> 3. If a and b are two non-empty and finite classes having no element in common, then the number of elements of the set of the two classes a and b is equal to the sum of the number of as and bs.[15]

In 3. above, Peano is stating, in modern set-theoretic notation, that if two sets a and b are disjoint, then the number of the elements of a *and* b is equal to the number of the elements of $a \cup b$. Peano, then, notes that the proposition holds even if one of the two classes, and even both, contain *infinite* elements; but now, we have, as a consequence, that:

$$x + \infty = \infty + x = \infty \tag{1}$$

[13][Data una classe a non sempre $num(a)$ è un N, poiché N non comprende né lo zero, né l'infinito (p. 101).]

[14][Il segno num è un segno d'operazione che ad ogni classe fa corrispondere o un N, o lo 0, o l'∞ (p. 102).]

[15][Essendo a e b due classi non nulle e finite, non aventi alcun individuo comune, allora il numero degli individui appartenenti all'insieme delle due classi a e b vale la somma dei numeri degli a e dei b (p. 102).]

where x is a finite quantity, and

$$\infty + \infty = \infty. \tag{2}$$

Immediately afterwards, he says: "*4. If the classes* a *and* b *are such that the second is contained in the first, and the class* b *is non-empty, and is not equal to* a*, and if the number of* a*s is finite, then the number of* b*s is also finite, and is less than the number of* a*s.*"[16]

Then he observes that: 'this proposition ceases to be valid if $num\ a = \infty$'.[17]

Propositions 3 and 4 also demonstrate that '∞' is taken by Peano to be different from Cantor's 'ω', since, as is known, by Cantor's conception, $\omega + n \neq n + \omega$. However, on the grounds of the arithmetical laws outlined in (1) and (2), one could be tempted to view '∞' as being equivalent to '$\aleph_0$'. But this would be a hasty conclusion. In section 4.3, we shall see that Peano's own intepretation of his 'infinitary numbers' does not automatically sanction the equivalence between his '∞' and '$\aleph_0$'.

One further comment is in order. Proposition 4 states that infinite classes could contain *proper* infinite subclasses. This implies, among other things, that 'standard' part-whole relationships cease to be valid in the infinite, a fact which, in turn, may have had some bearings on the development of Peano's ideas.

16[Se delle classi a e b la seconda è contenuta nella prima, e la classe b non è nulla, e non è eguale ad a, e se il numero degli a è finito, allora anche il numero dei b è finito, ed è minore del numero degli a (p. 101).]

17[Questa proposizione cessa di esistere se $num\ a = \infty$ (p. 101).]

3.2 The Cantor-Peano Correspondence

As already mentioned, Peano's early conception of the infinite is gradually modified, and then, by 1899, definitively replaced by Cantor's theory of the transfinite. What, ultimately, led Peano to change his mind remains an open question. Although the evidence is insufficient, we conjecture that the correspondence between Peano and Cantor, in the year 1895, proved instrumental in that respect.

In what follows, we review salient parts of the discussion Peano entertained with Cantor that are relevant to our purposes; as already noted, one side of the correspondence (from Peano to Cantor) is not extant, so Peano's comments and answers can only be, very approximately, deduced from the responses of his German colleague.[18] Cantor's letters cover the following subjects:

1. Cantor's articles that Peano intends to publish in his *Rivista Matematica*;
2. Cantor's distaste with Veronese's theory of actual infinitesimals;
3. Peano's request for explanations about Cantor's elusive definition of 'cardinal number'

In particular, (2) and (3) are fundamental, in our view, for the evolution of Peano's ideas on the infinite. Let's examine them in more detail.

[18] Cantor's letters to Peano examined in the present work are all collected in the Meschkowski edition of Cantor's letters, (Cantor, 1990), pp. 359ff. (n. 143-147). An overview of the contents of the Cantor-Peano correspondence is in (Kennedy, 1980), pp. 87-90.

As far as (2) is concerned, in the two letters of, respectively, July 27 and July 28, 1895, the discussion focuses on the conflict between Cantor's and Veronese's theories. On the German mathematician's view, what Veronese called 'ordered groups' were just a plagiarism of Cantor's 'simply ordered sets'. But then, Cantor points out what he thinks is the mistake that Veronese has made in trying to automatically extend the arithmetic of natural numbers to infinite numbers. As Cantor observes, in two passages of the July 27 letter:

> But if it is correct that his $\infty_1 = \omega +^* \omega$, his assertion that:
>
> $$2 \cdot \infty_1 = \infty_1 \cdot 2$$
>
> must be *incorrect*! [...] Anyway, his [Veronese's, *our note*] 'infinite numbers' seem tenable to me only if they are identified with some of my 'transfinite order-types'. In this case, however, they lack the law of commutativity for addition and multiplication (in general) on which he [Veronese, *our note*] lays so much stress.[19]

We do not know the content of Peano's answer; in any case, on Cantor's own impulse (28 July, 1895 letter), the 27 July letter was published by Peano in the

[19][Jedenfalls scheinen mir seine "unendlichen Zahlen" *nur dann haltbar, wenn sie mit gewißen von meinen "transfiniten Ordnungstypen" identificirt werden.* In diesem Falle fehlt ihnen aber das Gesetz der Commutabilität bei der Addition und Multiplication (im Allgemeinen), *worauf er solchen Nachdruck legt* ((Cantor, 1990), p. 360)]. The English translation is ours.

Rivista di Matematica.[20] So, it is imaginable that Cantor had managed to convince Peano of the correctness of his arguments.

As far as (3) is concerned, in the subsequent letters, the conversation between the two mathematicians turns to consider Peano's qualms about Cantor's definition of 'cardinal number'. This can be safely deduced from Cantor's September 14, 1895 letter, in which Cantor tries to clarify some notions contained in section 5 of (Cantor, 1895). From what Cantor says, it is clear that Peano was unsure about Cantor's notion of 'finite cardinal number' and about how Cantor introduced the induction principle, and asked his colleague for clarifications.

In the 21 September, 1895 letter, Cantor quotes a sentence written by Peano in his response to Cantor's previous letter: '*Where can one find the definition of finite cardinal numbers?*',[21] a clear sign that Cantor's previous letter was not entirely clarificatory, or, in any case, that Peano found Cantor's definitions unconvincing.[22]

Therefore, what seems to be likely is that, as a consequence of the clarifications Cantor gave Peano (and possibly also of Cantor's forceful rejection of Veronese's theory?), Peano started considering Cantor's transfinite the only correct conception of the infinite, and Cantor's theory of cardinal numbers pretty much definitive.[23] If in 1891 he had chosen not to fully adhere to Cantor's theory - or

[20]In the journal's August issue.

[21][Où est ce que l'on trouve la definition der endlichen Cardinalzahlen? ((Cantor, 1990), p. 365)].

[22]We thank an anonymous reviewer for drawing our attention to this possibility.

[23]However, Peano had already independently attempted to dash Veronese's conception in

simply not to explore it in depth - now he had further material at hand, including Cantor's clarificatory statements, which could reorient his views, a fact which would soon reflect on the *Formulaire*, where the theory of the transfinite is gradually prioritised.

3.3 The Infinite in the *Formulaire*

We finally turn to survey, very briefly, the modifications of Peano's conception as can be found in the volumes (editions) of Peano's *Formulaire de mathématiques*.[24] As said at the beginning, these have already been briefly taken into account by Paolo Mancosu in (Mancosu, 2016);[25] what follows aims to expand on, and complement, Mancosu's account.

The evolution of Peano's ideas on the infinite in the *Formulaire* spans a period of 4 years, and ends in 1899, when all traces of Peano's early conception disappear, and Cantor's transfinite ultimately takes over.

The first volume ((Peano, 1895)) saw the collaboration of other Italian mathematicians, such as G. Vailati, F. Castellano, G. Vivanti, R. Bettazzi. Of particular importance for our purposes are sections V and VI, written,

(Peano, 1892a) and (Peano, 1892b) (but see our comments in section 4.3). For Peano's assessment of Veronese's theory, see (Fisher, 1994) and (Cantù, 1999).

[24]All the volumes of the *Formulaire* appeared as supplements to issues of the *Rivista di Matematica* between 1895 and 1908. For an overview of the plan and evolution of this work, see the exhaustive (Cassina, 1955).

[25]See fn. 5.

respectively, by Peano and Giulio Vivanti; the latter, among other things, had taken part in the debate with Bettazzi on the infinitesimals hosted by the *Rivista* in the years 1891-1892.[26]

In section V of (Peano, 1895), entitled 'Classes de nombres', we find again the mentioning of just one infinite cardinality, '∞', which is now explicitly defined as one of the possible values of $num\ u$, where u is a class, as follows:[27]

$$4.\ num\ u = \infty \leftrightarrow num\ u \notin \mathbb{N}_0$$

Then Peano explains that $num\ u$ can take only three values:

$$5.\ num\ u = n \in \mathbb{N} \vee 0 \vee \infty$$

Moreover, '∞', in definition 6, is, again, characterised as enjoying *commutativity*:

$$6.\ a \in \mathbb{N}_0, \quad a + \infty = \infty + a = \infty + \infty = \infty \quad and \quad a < \infty$$

On the other hand, section VI, due to Vivanti, not to Peano, introduces, and uses, the symbols '*Nc*' and '*Ntransf*', denoting, respectively, Cantor's cardinals and ordinals, and deals with the corresponding set-theoretic notions. Therefore, in 1895, Peano did not entertain ideas much different from those appeared in (Peano, 1891), i.e. he did not subscribe to Cantor's views, while (Peano, 1895)'s

[26]The debate is accurately reconstructed in (Ehrlich, 2006), pp. 75-101.

[27]In the text of formulas (4)-(6) we have, for the sake of simplicity, chosen not to adopt Peano's original notation, but the modern one.

section VI, composed by Vivanti, which mentions Cantor's cardinals and ordinals, has no immediate connection with the one composed by Peano. This is consistent with our hypothesis that the correspondence with Cantor may have led Peano to change his mind after the publication of (Peano, 1895), which, in any case, must have been drafted well before 1895.

The second volume of the *Formulaire* consists of three parts, each published one year apart from the other, following the editorial plan below:

- §1, *Logique mathématique* (Peano, 1897);
- §2, *Arithmétique* (Peano, 1898)
- §3 (untitled) (Peano, 1899)

(Peano, 1898) contains the first manifestation of Cantor's theory in Peano's writings. Peano says:

> Here we define another idea, indicated by the sign Nc' similar to the previous one, indicated by num, but not identical. This definition is expressed by the only signs of logic (§1); it is therefore independent of the primitive ideas $\mathbb{N}_0, +, 0$. We could start arithmetic here. P211 expresses the coincidence of the signs num and Nc' when one deals with finite classes. But we will have for example:
> $num\, \mathbb{N} = num\, \mathbb{R}^+$, because the classes $\mathbb{N}$ and $\mathbb{R}^+$ are both infinite;[28] but $Nc'\, \mathbb{N} < Nc'\, \mathbb{R}^+$, because the power of $\mathbb{N}$ is less than

[28]In the original text, Peano uses the symbols N and Q, to denote, respectively, the set of the

the power of $\mathbb{R}^+$. Mr. Cantor indicated the power of a by $\overline{\overline{a}}$ (See RdM. A. 1895 p.130[29]), a notation that cannot be adopted in the Formulaire. M. Vivanti in F_1 VII §2 P1 [section VI of (Peano, 1895), *our note*] has replaced it with $Nc'\ a$ "the cardinal number of a.".[30]

From the quote above, it seems clear that Peano is beginning to change his mind, and gradually converting to Cantor's theory, although he still keeps mentioning both conceptions: Nc', denoting Cantor's cardinal numbers, now appears alongside num as a symbol of 'cardinality'.

(Peano, 1899) marks one further transformation of Peano's conception. In that work, for the first time, Peano 'merges' the symbols num and Nc' into a unique, and new, symbol, 'Num'. The Num's are now nothing but Cantor's transfinite

natural and of the positive *real* numbers; in our translation, we have, for simplicity, replaced N and Q with the modern $\mathbb{N}$ and $\mathbb{R}^+$.

[29]The reference, here, is to (Peano, 1895), p. 130.

[30][Nous définissons ici une autre idée, indiquée par le signe "Nc'" semblable à la précédente, indiqueée par "num", mais non identique. Cette définition est exprimée par les seules signes de logique (§1); elle est donc indépendante des idées primitives $\mathbb{N}_0, +, 0$. On pourrait commencer ici l'Arithmétique. La P211 exprime la coincidence des signes num et Nc' lorsqu'il s'agit de classes finies. Mais on aura par exemple: $num\ \mathbb{N} = num\ \mathbb{R}^+$, car les classes $\mathbb{N}$ et $\mathbb{R}^+$ sont toutes les deux infinies; mais $Nc'\ \mathbb{N} < Nc'\ \mathbb{R}^+$, car la puissance des $\mathbb{N}$ est plus petite que la puissance des $\mathbb{R}^+$. M. Cantor a indiqué la puissance de a par $\overline{\overline{a}}$ (Cfr. RdM. a. 1895 p.130), notation qu'on ne peut pas adopter dans le Formulaire. M. Vivanti dans F_1 VII §2 P1 l'a remplacée par $Nc'\ a$ "le nombre cardinal des a". ((Peano, 1898), p. 39)].

cardinalities, an ‘expansion’ of Peano’s former num subtly revealed by the change of notation. From (Peano, 1899) onwards, thus, Peano seems to fully adhere to Cantor’s theory.

In the *Formulaire*’s third volume, (Peano, 1901), in the Num section, Peano, finally, explains:

> $Num'Cls$ means “the number of a class”. These numbers coincide with the [elements of] $\mathbb{N}_0$ for the finite classes; G. Cantor calls them “cardinal numbers”. In F 1895 the symbol “Nc” was introduced to represent them [again, a reference to Vivanti’s section VI of (Peano, 1895)].[31]

Thus, already by 1901, and then in the subsequent volumes of the *Formulaire* (1903, 1908), any significant distinction between Peano’s and Cantor’s conceptions disappears, and Peano’s early conception of the infinite is only a distant memory.

3.4 *Enter* Vivanti

We have seen that Giulio Vivanti was involved in the writing of the *Formulario*, and we have cursorily mentioned that he interacted with Bettazzi on the issue of the existence of actual infinitesimals.[32] Now, his role with respect to the

[31][$Num'Cls$ signifie “le nombre d’une classe”. Ces nombres coïncident avec les N_0 pour les classes finies; G. Cantor les appelle “nombres cardinaux”. Dans F1895 on a introduit le symbole “Nc” pour les représenter. ((Peano, 1901), p. 70)]

[32]Cf. fn. 26.

evolution of Peano's ideas might have been equally prominent. Indeed, one could even conjecture that the transformation of Peano's ideas was, directly or indirectly, also due to Vivanti's advice.

This claim seems to be supported by, at least, two main reasons. The first is that Vivanti was acknowledged to be, at the time, as one of the main experts of (Cantorian) set theory. Herbert Meschkowski sharply expresses this fact in a note following Cantor's 3 December, 1885 letter to Vivanti:

> The number of mathematicians who around 1885 had confronted with Cantor's doctrine was still very small; the young Italian Vivanti belonged to them. [...] Vivanti's works on set theory [...] had, moreover, contributed to make Cantor's theory known in Italy.[33] ((Cantor, 1990), p. 251)

Indeed, as explained by Meschkowski, Vivanti and Cantor corresponded with each other for a decade (1885-1895). As a main correspondent of Cantor's, and early practitioners of Cantorian set theory, Vivanti must have realised the potential, and overriding strength, of the Cantorian theory compared to other conceptions of the infinite.

In the context of the debate on the infinitesimals appeared in the *Rivista di Matematica*, Vivanti had, significantly, stood out as an advocate of Cantor's conception against Bettazzi. At the same time, as evidenced by (Vivanti, 1891) and (Vivanti, 1895), Vivanti never held the view that theories of infinitesimals

[33]The translation is ours.

were, in the least, inconsistent.

In their exchange on the transfinite, Vivanti and Cantor dealt with a very broad range of themes, spanning both mathematical and philosophical aspects of set theory, as, for instance, in Cantor's 1886 letter, which would, subsequently, become part of Cantor's *Mitteilungen zur Lehre vom Transfiniten*.[34]

In 1893, Vivanti insisted with Cantor that the latter's attempts to expel actual infinitesimals from mathematics were doomed to failure, as he argued that du Boys-Reymond's use of infinitely large and small quantities was perfectly consistent with the *concept* of *number*. Cantor was equally harsh in rebutting Vivanti's comments.[35]

Coming to the fruits of Vivanti's lasting collaboration with Peano, this included an early 'Teoria di gruppi di punti', a work on *point sets* whose publication preceded that of the *Formulario*'s first edition. Moreover, Vivanti had already reviewed (Peano, 1892a) after its publication and, using his wide expertise of set theory, corrected Peano on a few points.[36] Subsequently, as we know, Vivanti was entrusted by Peano of the writing of the section on the 'Teoria degli insiemi' ('Set theory') for the 1895 version of the *Formulario*.

What is also worth mentioning is that Vivanti actively fostered the exchange of letters between Cantor and Peano, as can be gleaned, for instance, by the 1893 letter from Cantor to Vivanti which would then be published in the *Rivista di*

[34](Cantor, 1887), in (Cantor, 1932), pp. 409-411.

[35]Cf. Cantor's 13.12.1893 letter to Vivanti (fac-simile) reproduced in (Cantor, 1990), pp. 514ff.

[36]Cf. §4.3.

Matematica.[37] The correspondence between Peano and Cantor, as we already know, started two years later, in 1895.[38]

To sum up, it is plausible to conjecture that Peano's main 'set-theoretic' collaborator, Vivanti, sought to steer Peano's ideas, so to speak, towards the set-theoretic conception. In particular, Vivanti may not only have been the main connection between Peano and Cantor, but also between Peano and a (more) correct understanding of set theory.

4 Interpretations of Peano's Conception

In this section, we turn to examine the issue of what may have motivated Peano's early conception of the infinite, and discuss further textual sources which might help us better understand the evolution of Peano's ideas.

We will consider three main potential motivations: (1) the first is that, like Galileo before him, Peano was not fully convinced of the correctness of the 'bijection method' *vis-à-vis* Euclid's Axiom (the Part-Whole Principle); (2) the second, also of a markedly Galilean character, is that Peano's work on space-filling curves may have suggested to him that Cantor's notion of *cardinality* was not suitable for all mathematical contexts; 3) a third possible motivation is that Peano assumed that (the bulk of) Cantor's theory of the transfinite could be used to suit his own purposes, or even that Cantor's theory was consistent with his own conception.

[37]Cf. (Cantor, 1990), p. 505.

[38](Kennedy, 1980), p. 87-90.

We examine, respectively, (1) in §4.1, (2) in §4.2, and (3) in §4.3.

4.1 Between Galileo and Euclid

In his *Dialogues Concerning Two New Sciences* (1638), Galileo came to express skepticism about the possibility of comparing the sizes of infinite 'collections'. In the work's *First Day*, Galileo's spokesperson, Salviati, proposes to compare the sizes of infinite collections of objects by checking that each object in one collection is *uniquely* matched by an object in the other one. More specifically, Salviati proposes to compare *square* with *natural* numbers through a 'mapping function', $n \mapsto n^2$, which associates to each natural number its square number:

$$0 \leftrightarrow 0$$

$$1 \leftrightarrow 1$$

$$2 \leftrightarrow 4$$

$$3 \leftrightarrow 9$$

$$.. \leftrightarrow ..$$

Salviati and his interlocutor, Sagredo, agree that, using this method, it must be concluded that there are *as many* natural numbers *as* square numbers (that is, that $\mathfrak{s}(\mathbb{N}) = \mathfrak{s}(\mathbb{S})$, where $\mathfrak{s}(X)$ means 'size of X'). But now the dire puzzle unfolds in front of their eyes: it seems to be an 'established fact', commonsense knowledge, that there are *more* natural *than* square numbers, since $\mathbb{S}$ has lots of gaps 'in between', that $\mathbb{N}$ does not have; this is natural, since $\mathbb{S}$ is a *proper part* of $\mathbb{N}$ (i.e., using the set-theoretic notation, $\mathbb{S} \subset \mathbb{N}$). So, now relying on intuitions referring to the 'density' of $\mathbb{S}$ in $\mathbb{N}$, Salviati suggests that one should, in fact, more correctly, conclude that $\mathfrak{s}(\mathbb{N}) > \mathfrak{s}(\mathbb{S})$. This is what has come to be known as Galileo's Paradox.

Now, asks Salviati, which of the two horns of the dilemma is correct? Neither, apparently, as Galileo ultimately points out, since:

> This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another. ((Galileo, 1638), p. 31)

Now, Galileo does not go so far as to say that, but, since one cannot refer to one size as being 'greater than', 'less then', or 'equal to' another one in the *infinite*, it would, in principle, be fully legitimate and consequential to think that, to all purposes and intents, there is but *one* infinite cardinality. Thus, what we would have at hand, in (Galileo, 1638), would be the enunciation of an early 'single-cardinality' conception of the infinite dictated by the impossibility to come to terms with a conflict between two ways of counting in the infinite, one based on 'bijections' and one on 'densities'. The former method is formally enshrined in what would, later, become the central pillar of set theory, that is:

Cantor's Principle (CP). *Given two sets A and B, if there exists $f : A \to B$ which is 1-1 and onto (i.e., a bijection between A and B), then $\mathfrak{s}(A) = \mathfrak{s}(B)$.*[39]

The latter method upholds what was already known in antiquity as Euclid's Axiom (Common Notion V of the *Elements*), that is, the:

Part-Whole Principle (PWP). *Given two sets A and B, if $A \subset B$, then $\mathfrak{s}(A) < \mathfrak{s}(B)$.*

In our era, the 'set-theoretic era', as it were, one tends to think straight away that CP is the correct way to measure infinite collections, and that, as a consequence, Galileo's Paradox is no paradox at all, but rather the hallmark of the infinite itself, as also expressed by the notion of:

Dedekind-Infiniteness. A set is *infinite* if and only if it can be put in a *one-to-one correspondence* with a *proper* part of itself; otherwise, it is *finite*.[40]

But the correctness of PWP, as is known, was the prevailing view for many centuries, and still lay almost uncontested in Galileo's time.[41] Moreover, oscillations between compliance with CP or PWP could even be found, centuries later, in the work of avowed supporters of the bijection method like Bolzano.[42] Now, one could speculate that: 1) Peano shared Galileo's concerns about the

[39]CP may also be seen as a 'variant' of Hume's Principle mentioned in §1 (see fn. 4).

[40]The notion was first formulated in (Dedekind, 1888), V, 64.

[41]For a careful excursus of the history of several cardinality principles, we refer the reader to (Mancosu, 2009).

[42]Cf. (Mancosu, 2016), pp. 130-1.

possibility of articulating a theory of *distinct* infinite cardinalities, and that: 2) he settled on a sober 'one-cardinality' conception precisely because he was hesitant to choose among one of the two (known) methods of counting in the infinite.

In particular, Peano's 1891 conception might have reflected dissatisfaction both with a purely Cantorian *and* with the Euclidean point of view. This is (indirectly) proved by a remark made by Peano in a footnote of his (Peano, 1891), concerning Rodolfo Bettazzi's use of the bijection method to characterise the concept of *number* in (Bettazzi, 1887).[43] Bettazzi thought that, if two sets A and B could be put in a one-to-one correspondence, then any correspondence between them would have to be one-to-one. For the sake of our exposition, let us restate this as:

Bettazzi's Principle. *Given two sets A and B, if there exists a bijective function between A and B, then there is no function between A and B which is injective, but not surjective.*

Peano correctly objects to Bettazzi that one can biject infinite sets with their own proper infinite parts, that is, one can have *injections* of sets with themselves which, clearly, are not *surjective*. Further reflection on this fact might have led Peano to doubt that the bijection method could yield a satisfactory characterisation of the notion of (infinite) number, as envisaged by Bettazzi, and the most natural consequence of this would have been that he would not allow for the existence of different sizes in the infinite.

On the other hand, and regardless of his assessment of Bettazzi's Principle, if

[43]The passage is cited (and translated to English) by (Mancosu, 2016), p. 164-5.

Peano had wanted to stick with PWP, then he would have had to allow for different infinite sizes anyway, e.g., that of the *odd* numbers and of $\mathbb{N}$ itself, but this is inconsistent with his conception as we know it.

Overall, Peano's hesitancy between Euclid and Cantor, so to speak, nails down the 'Galilean' character of his standpoint, which could be summarised as follows: there is no way to provide scope for different infinite sizes in a way which is consistent with both the Cantorian and the Euclidean way of counting in the infinite, and, as a consequence, one has no other choice but prudently retreat to a single-cardinality conception of the infinite.

4.2 Equinumerosity and Continuity

One further reason for skepticism about the 'measurability' of the infinite could have potentially been suggested to Peano by his work on *space-filling curves*, such as his own curve (Fig. 1). Let us see how in more detail.

(Cantor, 1878) discussed groundbreaking results on bijective correspondences between points sets in different *dimensions*. In particular, Cantor proved the bijectability of the closed interval $I = [0, 1]$ with the unit square $I^2 = [0, 1] \times [0, 1]$. The following year, Eugen Netto proved that such a mapping, however, cannot be a *continuous* function.[44] In 1890, in a paper appeared in the *Mathematiche Annalen*, and which is certainly among the Piedmontese mathematician's best-known contributions, Peano found a suitable geometrical

[44]Intuitively, a continuous function can be thought of as one which could be 'drawn with a free movement of the hand'.

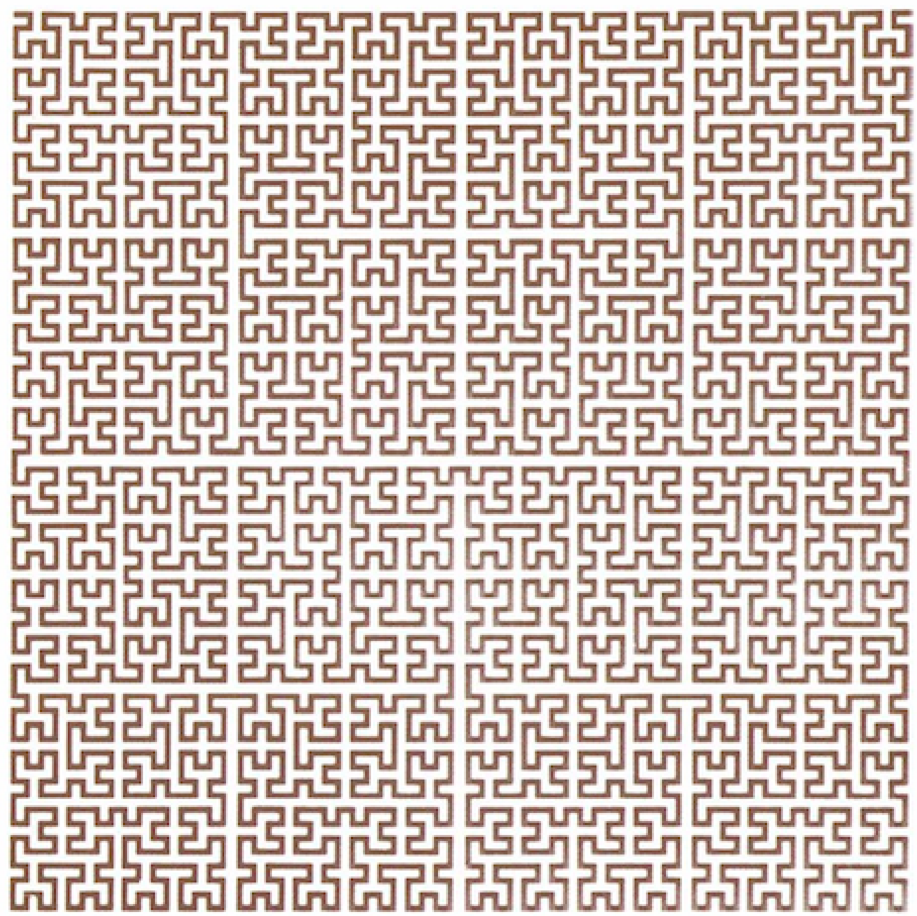

Figure 1: A representation of the Peano curve (from (Kennedy, 1980) p. 48)

expression of Netto's result in his curve.[45]

The curve is an example of a *continuous* correspondence between points on the interval I and the unit square I^2, which is not *bijective*, since it is only *surjective*. As Peano shows, for each pair of numbers (x, y), corresponding to a point of the unit square I^2, there exists at least one number t of I whose image is precisely the

[45](Peano, 1890), also in (Peano, 1957), pp. 110-14.

point under consideration, i.e. I densely fills the entire unit square, but there exist distinct elements of I with which the same pair of numbers (x, y) is associated. In particular, the same pair of numbers (x, y) may be associated with *one*, *two* or *four* distinct values of I, depending on the construction of x and y in terms of the expansion of powers of 3.

The Peano curve is a quintessential example of the counterintuitive aspects of the geometrical infinite. (Cantor, 1878)'s proof had, already quite unexpectedly, demonstrated that the *dimension* of a point set did not matter as far as its *cardinality* was concerned (since $card(I) = card(I^2)$ in the aforementioned example), whereas the Peano curve shows that any continuous mapping between I and I^2 would rather suggest that $\mathfrak{s}(I^2) \leq \mathfrak{s}(I)$, if such a non-Cantorian notion of 'size' were really available.

As a consequence, in his curve, Peano may have found one further mathematical example of the practical 'defectiveness' of the Cantorian notion of infinite cardinality, and it is very tempting to see this as one more reason for Peano to question the value of that notion, along the lines of the 'Galilean' standpoint we have sketched in the previous subsection.

4.3 (Geometric) Infinitesimals and Cantorian Set Theory

(Peano, 1892a) is an attempt to improve on Cantor's earlier proof that actual infinitesimal segments are *inconsistent*.[46] In the article, Peano shows a more than decent understanding of Cantorian set theory, but also seems to want to use it,

[46] A detailed analysis of Cantor's proof (for which see (Cantor, 1887)) is carried out in (Ehrlich,

very originally, in a way compatible with his 'single-cardinality' conception of the infinite.

Key notions in that work are those of 'bounded' and 'infinitesimal segment'. Given the half-line with origin in o, a 'bounded' segment is a segment op with origin in o and end in p. Peano also denotes a bounded segment with u, or with $\overline{ou}$. An 'infinitesimal segment' is an $\overline{ou}$ lying inside a bounded segment v denoted with $u = v/\infty$. u's full definition reads as follows:

> We say that the *segment* u is infinitesimal with respect to v and we write $u \mathrel{\epsilon} v/\infty$, if every multiple of u is less than v [...].[47]

Then Peano proceeds to define the multiple of infinite order of u, '∞u'. He says:

> We shall posit:
>
> $$\infty u = \bigcup \mathbb{N} u$$
>
> that is, we call multiple of u of infinite order the set of points which either lie on some segment u, $2u$, $3u$, ..., or the upper bound of the multiples of u.[48]

2006), pp. 27-51.

[47][Dicesi che il segmento u è infinitesimo rispetto al segmento v, e scriveremo $u \mathrel{\epsilon} v/\infty$, se ogni multiplo di u è minore di v [...] (p. 113)] Note the use of Peano's ϵ symbol, which means: 'is'. Page numbers for (Peano, 1892a) are, again, those of Cassina's edition, for which see fn. 12.

[48][Porremo

$$\infty u = \bigcup \mathbb{N} u$$

Now, Peano asserts that, by the definition of infinitesimal, if u is infinitesimal, also '∞u' must lie *inside* v, and so must all other infinitary multiples of u. In a crucial passage of (Peano, 1892a), he says:

> We can add ∞ to itself, thus obtaining $2\infty u$, and, generally, we can form all multiples of ∞u; we can multiply ∞u by ∞, and obtain $\infty^2 u$ and so on. But all these various segments, which one obtains multiplying u by Cantor's transfinite numbers are all equal to one another [...].[49]

As a consequence, infinitesimal segments, if such things existed, would blatantly violate the properties of bounded segments, which require that, for instance, a segment of length $2\infty u$ be *greater than* ∞u. Hence, Peano concludes, infinitesimal segments are inconsistent with 'standard' representations of the geometrical space.

Leaving aside entirely the issue of the correctness of Peano's argument, what is most relevant to our purposes is Peano's peculiar use of Cantor's transfinite numbers and arithmetic. This is indeed puzzling, as, in Cantor's theory, as we

---

cioè chiamiamo multiplo d'ordine infinito di u l'insieme dei punti che stanno sopra qualcuno dei segmenti u, $2u$, $3u$, ..., o il limite superiore dei multipli di u. (p. 113)] For the sake of simplicity, in the formula, we have chosen to replace Peano's original notation: $\cup' Nu$ with $\bigcup \mathbb{N}u$.

[49][Possiamo sommare ∞u con sé stesso, ottenendo così $2\infty u$, ed in generale possiamo formare tutti i multipli di ∞u; possiamo moltiplicare ∞u per ∞, ed ottenere $\infty^2 u$ e così via. Ma tutti questi varii segmenti, che si ottengono moltiplicando u pei numeri transfiniti di Cantor sono eguali fra loro, [...]] (p. 113)

know, $\omega + 1 > \omega$, thus, $\omega + 1 \cdot u$ is, contrary to what asserted by Peano, greater than $\omega \cdot u$.

Indeed, Giuseppe Veronese was immediately able to raise precisely this issue. In (Veronese, 1892), which is meant to be a response to (Peano, 1892a), the Italian mathematician correctly diagnoses what seems to be the trouble with Peano's proof:

> But these equalities [$\infty u = 2\infty u$ = $\infty^2 u = ...$, *our note*] do not depend on the properties of Mr. Cantor's transfinite numbers, for which hold: $\omega + 1 > \omega$, $2\omega > \omega$, etc., but precisely on considering ∞u as *unlimited* (our italics).[50]

In the passage, Veronese sharply points out that it is Peano's own definitions of ∞u, $2\infty u$, etc., not Cantor's *transfinite arithmetic*, which allow Peano to assert that these quantities are all equal. Similar remarks were made by Giulio Vivanti, who, in his (Vivanti, 1895), explained why the argument was doomed to failure. Vivanti argued that Peano viewed $(\infty + 1)u$ as the upper bound of $(n + 1)u$, but now the points of $(\infty + 1)u$ must lie either on one of the finite multiples of u, or be $(\infty + 1)u$ itself. So, $(\infty + 1)u$ is precisely the *same* segment as ∞u; hence, $(\infty + 1)u$ and ∞u must be *equal*. But, as Vivanti points out, if it is true that Cantor's ω is the limit of n, it is also true that $\omega + 1$ is not the *limit* of any number, and, as already pointed out by Veronese, $\omega + 1 > \omega$, and $(\omega + 1)u > \omega u$.[51]

[50](Veronese, 1892), p. 74.

[51](Vivanti, 1895), p. 69.

So, Peano's reliance upon Cantor's transfinite numbers in his proof does not seem to serve very well his purpose of showing that geometrical infinitesimals are inconsistent. In order to rescue the force of the argument, (Freguglia, 2021) conjectures that the reason why Peano holds that:

$$\infty u = 2\infty u = \infty^2 u = ...$$

is the fact that Peano 'assimilates' '∞' to '$\aleph_0$'.[52] This makes sense because, Freguglia argues, while, as said, $\omega \neq \omega + 1$, on the contrary, $\aleph_0 = \aleph_0 + 1$ and, in general, if κ is a transfinite *cardinal* number, and n a natural number:

$$\kappa + n = \kappa \cdot n = \kappa^n = \kappa$$

Freguglia's interpretation has some merit, but seems to be missing, at large, the mark. In particular, it seems to be rather unjustified to force this interpretation on Peano's multiple references to Cantor's transfinite numbers; the numbers Peano is referring to here are, it seems to us, and as also understood by Veronese and Vivanti, just Cantor's *transfinite ordinals* ($\omega, \omega + 1, ..., \omega + \omega$, ...).

However, following the spirit, but not the letter, of Freguglia's intepretation, one could say that Peano may have taken his numbers (∞u, $\infty^2 u$, ...) to be equivalent to $\aleph_0$ in the sense that he thought that they were, like Cantor's ordinals, *linearly ordered*, but that they had, nonetheless, the same *cardinality* (since they are all

[52]Cf. (Freguglia, 2021), p. 152ff. The author says: 'Peano explicitly assimilates ∞ to $\aleph_0$ [...]' and then explains ibid., fn. 14: '[i]n the sense that it has the same arithmetic behaviour.'

countable). By this interpretation, since what counts as *length* of a geometrical segment is the measure expressed by a *cardinal* number, one must conclude that segments of length ∞u, $\infty^2 u$ cannot but be *equal*. But there's a problem with this interpretation: Cantor's ordinals may also be *uncountable*, that is, they may have different cardinalities, and Peano might have been aware of this.
We wish to put forward one last interpretation of Peano's argument, which aligns more with the content of Vivanti's and Veronese's comments above. By this interpretation, Peano viewed his 'infinite numbers' as being all equal to each other not because he thought that '∞' was equivalent to '$\aleph_0$', but because he thought that there was just *one* infinite cardinality. So, when Peano is referring to 'Cantor's numbers' in the aforementioned passage, he is really likening his own numbers to Cantor's ordinals (as supposed by Veronese and Vivanti), but he is also committing himself to the view that those numbers must all have the same cardinality, because, by his own conception, there exists *only* one, '∞', which cannot be transcended. Thus, eventually, Peano's use of 'Cantor's numbers' in the proof could be seen as an original attempt to merge his own, 'single-cardinality', conception with Cantor's transfinite.

To sum up, the three motivations we have examined, overall, seem to support the following conclusion: at least in the years 1891-1895, Peano advocated a carefully thought of, and original, conception of the infinite, which he believed was motivated by several practical mathematical contexts (such as those examined in (Peano, 1890) and (Peano, 1892a)) and reasons. The fact that he, subsequently, abandoned this conception does not mean that he thought it to be faulty, 'half scarce made up' in any respect, but only that he gradually came to acknowledge that Cantor's theory was more general, far-reaching and, ultimately, fruitful than his own.

5 Concluding Summary

We have reviewed the development of Peano's conception of the infinite as far as the publication of the third volume of his *Formulario* in 1901. We have seen that, already by the publication of the third section of the second volume in 1899, Peano's conception had aligned with the, then already 'meainstream', set-theoretic conception. In our view, both Cantor and Vivanti may have played a role in persuading Peano to abandon his earlier view.

As far as the view itself is concerned, it seems to us to have adequately shown that Peano had perfectly clear in his mind its consequences, the way it could be properly used in mathematical contexts, and how it related to, and could even merge with, aspects of Cantor's transfinite.

Moreover, we have seen that, until his full set-theoretic 'conversion', Peano might have been very doubtful of the efficacy, and adequacy, of the bijection method, especially insofar as this conflicted with PWP.

Finally, we have conjectured that these doubts might have led him to support a 'single-infinite-cardinality conception' which partly incorporates, and validates, Galileo's skeptical view about the measurability of the actual infinite.

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