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a cura di
Corrado Giarratana

RUBBETTINO

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Claudio Ternullo

Peano's Proof of the Inconsistency of Infinitesimals and Set Theory

1. *The debate on infinitesimals in the late XIX century and set theory*

With the development of modern analysis and the introduction of methods based on the concept of *limit* that entirely dispensed with infinitely small quantities, it seemed that the analysis of Newton and Leibniz based on actual infinitesimals had been put aside once for all. The field of the *reals* was definitively fixed in the form of an *Archimedean field*, i.e. a field in which there are *no* infinitesimals and *no* infinities[1].

In fact, the study of non-Archimedean fields continued well into the 19th century. For example, Paul du Bois-Reymond, in France, and Otto Stolz, in Germany, introduced the notion of *order of infinity* of a function, and thus managed to define a *non-Archimedean field of functions* that could be infinitesimal with respect to each other.

The 19th century infinitesimalists committed themselves to defending the introduction of infinitely small quantities also from a practical point of view[2]. Du Bois-Reymond, for example, says: «the infinitely small is a mathematical quantity that has properties also shared by finite quantities and despite doubts and perplexities,

[1] In particular, the axioms for the reals are the standard field axioms with, in addition, the Cantor-Dedekind axiom of continuity, for which see later in the text.

[2] See, in particular, O. Stolz, *Die unendlich kleinen Grössen*, in «Berichte des Naturwissenschaftlich-Medizinischen Vereines in Innsbruck», n. 14, 1884, pp. 21-43.

infinitely small quantities have the same right to exist as infinitely large ones»[3].

A few years later, the Italian mathematicians Giuseppe Veronese and Tullio Levi-Civita developed number systems that included not only ordinary *finite*, but also *infinitesimal* and *infinite* quantities.

For instance, Veronese introduced infinite quantities of different orders of magnitude: ∞_1, ∞_2, ... and then combined them with the real numbers, while Levi-Civita considered what he called *monosemii*, numbers of the form: $a\alpha^{\nu}$, with ν an integer, a a real number and α an infinite quantity. Both Veronese's and Levi-Civita's numbers gave rise to non-Archimedean fields of reals[4].

However, Veronese even took a step further, defining a line (the "line of Veronese") which contained infinite and infinitesimal *segments*, thus nailing down the reality of infinitesimals as *geometric* entities.

Veronese justified his approach on the grounds of a "plenitude principle" for mathematical entities, which I will call here, for the sake of my purposes:

> **Veronese's Principle (VP)**. Given a universe $\mathfrak{A}$ of mathematical entities, if it is still possible to *conceive of* objects outside $\mathfrak{A}$ in a consistent manner, then it is legitimate to take into consideration an extension $\mathfrak{A}'$ of the universe that contains also those objects[5].

Veronese then argued that, since it was possible to conceive of further entities (e.g., the *infinitesimals*) beyond the real numbers in a consistent manner, it was legitimate to consider an expansion $\mathfrak{R}'$ of the universe $\mathfrak{R}$ of the real numbers *with* infinitesimals, an approach

[3] P. Du Bois-Reymond, *Über die Paradoxien der Infinitär-Calcüls*, in «Mathematische Annalen», n. 11, 1877, pp. 150-167, in V. Benci, P. Freguglia, *La matematica e l'infinito*, Carocci, Roma 2019, p. 68. The English translation is mine.

[4] P. Ehrlich, *The absolute arithmetic continuum*, in «The Bulletin of Symbolic Logic», n. 18-1, 2012, pp. 1-45 contains an exhaustive examination of both conceptions.

[5] G. Veronese, *Fondamenti di geometria a più dimensioni e a più specie di unità rettilinee esposti in forma elementare*, Tipografia del Seminario, Padova 1891, pp. 13-14. For Veronese's infinitary conceptions, see G. Fisher, *Veronese's Non-Archimedean Linear Continuum,* in P. Ehrlich (ed. by), *Real Numbers, Generalizations of the Reals, and Theories of Continua*, Kluwer Academic Publishers, Dordrecht 1994, p. 107-146.

that would later be fully made sense of by Abraham Robinson with the creation of non-standard analysis[6].

In general, mathematicians endeavouring to resurrect infinitesimals placed a lot of emphasis on the fact that these arose from a "natural expansion" of the Archimedean field of reals, insofar as also infinitesimals satisfied all the axioms of the ordered field of the reals, except for, as is clear, the:

Axiom of Continuity (Dedekind-Cantor). Given any two infinite s equences α^n and b^n of real numbers such that, $\forall n \in \mathbb{N}$, $a_n < b_n$, there exists a unique $\xi \in \mathbb{R}$, such that $a_n \leq \xi \leq b_n$.

It is noteworthy, and somewhat ironic, at the same time, that an argumentative strategy based on an "expansion" of the concept of number (of "integer") had been used by Cantor in order to justify the introduction of his own transfinite numbers, ordinals (ω, $\omega + 1$,... $\omega + \omega$, ..) and cardinals ($\aleph_0$, $\aleph_1$ ...,)[7]. But then, it was Cantor himself who stood out as the most radical opponent of an expansion of the number domain which incorporated infinitesimals.

The philosophical debate between those, like Veronese, who supported infinitesimals (even geometric infinitesimals), and those, like Cantor, who opposed them, became, at traits, very heated, as it gradually involved some of the most eminent logicians and mathematicians of the time (among others, Peano, Russell, Frege, Hilbert and others)[8].

Cantor expressed his distaste with what he believed to be "monstrous numbers", and, in an 1893 letter to the Italian mathematician Giulio Vivanti, even compared infinitesimals to a "cholera bacillus" that was infecting mathematics[9].

[6] Cf. A. ROBINSON, *Non-Standard Analysis*, North-Holland, Amsterdam 1966.

[7] G. CANTOR, *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen*, Teubner, Leipzig 1883, reproduced in ID., *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, Springer, Berlin 1932, pp. 165ff. See also Cantor's 1895 letter to Wilhelm Killing, later in the text. For a review of all fundamental set-theoretic notions, see J. BURGESS, *Set Theory*, Cambridge University Press, Cambridge 2022.

[8] Cf., among others, the recent and useful account in V. BENCI, P. FREGUGLIA, *op. cit.*, pp. 77-90 and the classic P. EHRLICH, *The Rise of Non-Archimedean Mathematics and the Roots of a Misconception I: the Emergence of Non-Archimedean Systems of Magnitudes*, in «Archive for the History of Exact Sciences», n. 60-1, 2006, pp. 1-121.

[9] Cf. J.W. DAUBEN, *Georg Cantor. His Mathematics and Philosophy of the Infinite*, Princeton University Press, Princeton (NJ), 1979, p. 131.

In an another letter to the German mathematician Wilhelm Killing, Cantor further explained: «Of his infinitely large numbers […] Veronese […] says that they are introduced starting from hypotheses different from mine. But mine do not depend absolutely on any hypothesis, but are immediately derivable from the concept of the whole. They are just as necessary and free of arbitrariness as finite integers»[10]. Cantor formulated several arguments, both mathematical and philosophical, against the use of actual infinitesimals[11]. Intuitively, he saw a fundamental disanalogy between the *transfinite* numbers and, for instance, Veronese's numbers: transfinite numbers were *sets*, whereas infinitesimal numbers were not grounded in set theory, so they could not have the same foundational status as transfinite ordinals.

Moreover, Cantor thought that infinitesimal numbers were dispensable. In his *Grundlagen*[12], he even seems, at traits, to avow some sort of indispensability argument:

Indispensability Argument (IA). All and only those mathematical entities that are fundamental to the development of mathematics should be taken to be legitimate.

Through an application of IA to infinitesimals, Cantor was now able to assert that these were *illegitimate*, insofar as they could not give any concrete contribution to the development of mathematics.

Peano intended to make the most of the Cantorian set-theoretic approach in his proof of the inconsistency of infinitesimals. However, it will be apparent in due course that Peano's use of set theory is only partially faithful to this approach (sections 2-3), as the Italian mathematician rather seemed to want to cling to a different conception of the infinite, the examination of which (section 4), in my view, allows one to make sense of his proof more accurately, and more sensibly, than other interpretations.

[10] *Ibidem*, p. 351.

[11] For a thorough discussion of Cantor's arguments, see, again, P. EHRLICH, *The Rise of Non-Archimedean Mathematics and the Roots of a Misconception I: the Emergence of Non-Archimedean Systems of Magnitudes,* in particular, sections 6-8.

[12] G. CANTOR, *Grundlagen einer allgemeinen Mannigfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen*, op. cit.

2. *Peano's Inconsistency Proof.*

Peano's proof is contained in an article appeared in the «Rivista di Matematica» in 1892[13]. At the very outset, Peano declares that he intends to improve on Cantor's proof, since the latter would be so «concise to be judged incomplete»[14].

In order to do this, Peano introduces the notion of *bounded* segment. Given the half-line with origin in *o*, a bounded segment is a segment *op* with origin in *o* and end in *p*. Peano also denotes a bounded segment with *u*, or with $\overline{ou}$, and I will conform to this use throughout.

A bounded segment has certain features that Peano lists very carefully. In particular, bounded segments can be *added* and *multiplied* following the basic arithmetical laws for addition and multiplication. For instance, the sum of a bounded segment *u* with itself will be a bounded segment *2u*, the multiple of a segment *u* by a certain natural number *n* will be the segment *nu*.

For the sake of his proof, Peano then proceeds to define an *infinite* multiple of *u*, ∞u. He says:

> We shall posit:
>
> $$\infty u = \cup \mathbb{N} u$$
>
> that is, we call multiple of *u* of infinite order the set of points which either lie on some segment *u*, *2u*, *3u*, ... or the upper bound of the multiples of *u*[15].

This notion may be further explained as follows. Assuming that an *infinite* geometric segment ∞u exists, this could be taken to be the *infinitary sum* of all segments *nu*, for all natural numbers *n*, that is,

[13] G. PEANO, *Dimostrazione dell'impossibilità di segmenti infinitesimi costanti*, in «Rivista di matematica», n. 2, 1892, pp. 58-62.

[14] *Ibidem*, p. 110. Page numbers of all Peano's quotes refer to the third volume of Giuseppe Cassina's edition of Peano's works, G. PEANO, *Opere scelte*, vol. 3, Cremonese, Roma 1959. The English translations are all mine.

[15] «Porremo

$$\infty u = \cup \mathbb{N} u$$

cioè chiamiamo multiplo d'ordine infinito di *u* l'insieme dei punti che stanno sopra qualcuno dei segmenti *u*, *2u*, *3u*, ... o il limite superiore dei multipli di *u*» (*ibidem*, p. 113). For the sake of simplicity, in the formula, I have chosen to replace Peano's original notation: $\cup$ *'Nu* with the more usual $\cup \mathbb{N} u$.

the *set* of all points lying in some *nu*, for all natural numbers *n*, and their *upper bound*.

Finally, Peano defines *infinitesimal* segments. He says: «We say that the *segment u* is infinitesimal with respect to *v* and we write $u \in v/\infty$, if every multiple of *u* is less than *v* […]»[16].

Hence, an infinitesimal segment *u* would be a bounded segment $\overline{ou}$, which:

1. is always defined with respect to some other finite bounded segment *v*
2. lies inside *v* and
3. is such that, for all natural numbers *n*, $nu<v$

It should be noted that the definition of an infinitesimal bounded segment just follows from the "standard" definition of an infinitesimal quantity ϵ, which requires that ϵ must be smaller than any *finite* quantity: so, properties 1)-3) of geometric infinitesimals are consistent with that definition.

Here's how Peano's argument unfolds. Suppose, for a contradiction, that an infinitesimal segment really exists. We can then imagine what the infinite bounded segment, with *u* infinitesimal, would be like. By both the definition of infinitesimal and that of infinite bounded segment, *∞u must also lie inside v*. This is because all the points of ∞u must lie on some *nu*, for some natural number *n*, and, by the definition of infinitesimal, each *nu* is less than *v*, so ∞u can't possibly be greater than *v*. This also extends to the upper bound of the points in ∞u, since if this lay beyond *v*, then there would be some *other* point of ∞u which would also lie beyond *v*, but this is impossible.

But then one could consider also further infinitary multiples of *u*, that is, segments whose length could be, for instance, $(\infty + 1)u$, always with *u* infinitesimal. The possibility of carrying out this process had just been granted by Cantor's set-theoretic methods, which allow one to count past infinity (past ω, in fact, in Cantor's notation). The result of this process would be as follows:

We can add ∞ to itself, thus obtaining $2\infty u$, and, generally, we can form all multiples of ∞u; we can multiply ∞u by ∞, and obtain $\infty^2 u$ and so on. But

[16] «Dicesi che il segmento *u* è infinitesimo rispetto al segmento *v*, e scriveremo $u \in v/\infty$, se ogni multiplo di *u* è minore di *v* (*ivi*, p. 113). Peano's ϵ means: 'is'.

all these various segments, which one obtains multiplying u by Cantor's transfinite numbers, are all equal to one another[17].

The reason why it has to be so is that, given an infinitary bounded segment different from ∞u, say, $2\infty u$, all of its points must lie in some $2nu$, for all natural numbers n, and, in addition, have $2\infty u$ as upper bound. So, in practice, such a segment would be equal to (indistinguishable from) the segment the points of which have ∞u as upper bound.

Now, Peano explains, a fundamental law of bounded segments implies that, e.g., $2u > u$, but we have just seen that $2\infty u = \infty u$. As a consequence, infinitesimal segments, if they existed, would violate a fundamental property of bounded segments. Hence, Peano concludes, infinitesimal segments are inconsistent with our conception of the geometric space, which concludes the proof.

In the proof, there are at least two inferential steps that, *prima facie*, seem unwarranted, and, to some extent, even wrong. One is the assumption that, given any transfinite ordinal Ω different from ∞, one always invariably obtains that $\Omega u = \infty u$, with u infinitesimal. The second one is that this, if correct, would be problematic.

I will just be concerned with the first issue, as the second one has already been satisfactorily examined in other, very authoritative works which have shown how Peano's, as well as Cantor's, to begin with, inconsistency proofs may be successfully defeated[18].

3. *Inconsistency Debunked: Veronese, Vivanti and Freguglia*

In order to address the issue, it is necessary to look into (often neglected) works which may be seen as a direct response to Peano's proof as well as at the examination of its content in a recent article by Paolo Freguglia[19].

[17] «Possiamo sommare ∞u con sé stesso, ottenendo così $2\infty u$, ed in generale possiamo formare tutti i multipli di ∞u; possiamo moltiplicare ∞u per ∞, ed ottenere ∞^2u e così via. Ma tutti questi varii segmenti, che si ottengono moltiplicando u pei numeri transfiniti di Cantor sono eguali fra loro» (*ivi*, p. 113).

[18] Cf. again, Paul Ehrlich's comprehensive works cited in footnotes 5 and 8.

[19] P. Freguglia, *Peano and the Debate on Infinitesimals*, in P. Cantù, E. Luciano (ed. by), *The Peano School. Logic, Epistemology and Didactics*, «Philosophia Scien-

But first, two technical remarks are in order. Peano, in the proof, always uses the symbol '∞' to denote infinite quantities, also when this symbol is explicitly construed by him as referring to Cantor's transfinite ordinals, for which the correct notation would be "ω". Although this seems to have little relevance, it already reveals Peano's intention to use Cantor's technical concepts quite freely.

The second remark is that, while Peano seems to be well acquainted with Cantor's theory, he also seems to hold that, for instance, $\omega + 1$ and ω are the "same infinite", whereas, as is clear, set-theoretically, $\omega + 1 > \omega$. What may be the reason for such a conspicuous misunderstanding of Cantorian set theory?

Giuseppe Veronese also raised this issue. In a review of Peano's article containing the "inconsistency proof", the Italian mathematician diagnoses what seems to be one of the main troubles with Peano's proof:

> But these equalities $\infty u = 2\infty u = \infty^2 u = ...$ do not depend on the properties of Mr. Cantor's transfinite numbers, for which holds: $\omega + 1 > \omega$, $2\omega > \omega$ etc., but precisely on considering ∞u as *unlimited* (my italics)[20].

In essence, Veronese explains that the reason why Peano needs to see all infinite quantities as being equal is that, for his proof to go through, ∞u should correspond to an absolute "unlimited" infinite, and not, contrary to what Peano himself asserts, to Cantor's ωu.

The Italian mathematician Giulio Vivanti, who had taken part in the debate with Cantor on the actual infinitesimals and was also well versed in set theory, also briefly commented on Peano's proof along the same lines as Veronese's.

Vivanti argued that, while it was true, on the grounds of the definitions of infinite multiple of a bounded segment and that of infinitesimal segment, that $(\infty + 1)u$, for instance, was the same segment as ∞u, on the contrary, $(\omega + 1)u > \omega u$, since, as is known, and as already pointed out by Veronese, $\omega + 1 > \omega$[21], and nowhere in the

tiae», n. 25-1, 2021, pp. 145–156.

[20] G. Veronese, *Osservazioni su una dimostrazione contro il segmento infinitesimo attuale*, in «Rendiconti del circolo matematico di Palermo», n. 4-1, 1892, p. 74.

[21] G. Vivanti *Review of G. Peano's "Dimostrazione dell'impossibilità di segmenti infinitesimi costanti"*, in «Jahrbuch über die Fortschritte der Mathematik», n. 24, 1895, p. 69.

proof is one given any compelling reason to assume that Peano's ∞u precisely corresponds to Cantor's ωu. On the contrary, as Veronese had remarked, there are places in Cantor's own proof of the inconsistency of infinitesimals where the German mathematician envisages the existence of a segment $(\omega + 1)u$ *greater than* the segment ωu, with u infinitesimal.

Let us now take into account Freguglia's interpretation. Freguglia states, from the outset, that the reason why Peano holds that:

$$\infty u = 2\infty u = \infty^2 u = \ldots$$

is that he believes that "∞" equates with "$\aleph_0$"[22]. Freguglia says: «Peano explicitly assimilates ∞ to $\aleph_0$ [...]», then explains *ibid*., fn. 14: «[i]n the sense that it has the same arithmetic behaviour». This makes full sense, Freguglia argues, because while, as said, $\omega \neq \omega + 1$, on the contrary, $\aleph_0 = \aleph_0 + 1$ and, in general, if κ is a transfinite *cardinal* number, and n a natural number:

$$\kappa + n = \kappa \cdot n = \kappa^n = \kappa$$

Freguglia's interpretation has some merit, but seems to be, at large, unwarranted on the grounds of the evidence to be found in Peano's other works, and Peano's level of understanding of Cantorian set theory.

The Cantorian numbers Peano is referring to in his proof are, it seems to me, and as understood by Veronese and Vivanti, just Cantor's *transfinite ordinals*:

$$\omega, \omega + 1, \ldots, \omega + n, \ldots, \omega + \omega, \ldots, \omega^2, \ldots$$

However, following the spirit, but not the letter, of Freguglia's interpretation, one could say that Peano may have considered his own numbers $\infty, \infty^2, \ldots$ to be equivalent to $\aleph_0$ not because he thought they did not behave like Cantor's ordinals, but because he essentially considered their *cardinality*, which is precisely $\aleph_0$ (since they are all *countable*). In particular, since what counts as *length* of a geometrical

[22] P. Freguglia, *Peano and the Debate on Infinitesimals*, cit., p. 152ff.

segment is the measure expressed by a *cardinal* number, one concludes that the segments ∞u, ..., $2\infty u$, ... $\infty^2 u$, ... are all equal, since they must have the same length. But there would still be one problem with this interpretation: Cantor's ordinals may also be *uncountable*, that is, they may have different cardinalities, and Peano might have been aware of this (although, admittedly, for his purposes he may have just thought that the countable ones were sufficient).

To sum up, through examining Veronese's, Vivanti's and Freguglia's construals of Peano's proof, what emerges is that Peano may have taken different ordinals to be the same infinite, since either he considered Cantor's transfinite ordinals under the mere aspect of cardinality or because he forced on them the view that the infinite is just one "unlimited" quantity, as stated by Veronese, and as required by his definition of ∞u.

In the next section, I will state my own interpretation which differs from all the ones examined so far, and is based on work I have done elsewhere on Peano's conception of the infinite.

4. *Peano's "Single-Cardinality Conception" and the Inconsistency Proof*

I will now proceed to expound what, in a recent article of mine, I have labelled Peano's "single-cardinality" conception of the infinite[23].

In 1891, two years after *Arithmetices principia, nova methodo exposita*[24], Peano publishes his equally famous article *Sul concetto di numero* on the «Rivista di matematica», the journal he had founded the same year[25].

In section 9 of that article, Peano defines a function, , whose domain consists of "classes" (denoted *a*, *b*, *c*, ..., *u*, ...), and whose values are the "cardinalities" of these classes; in Peano's own words, *num a* is the number of elements of the class *a*[26]. Now, if is a finite class, then

[23] C. Ternullo, I. Fascitiello, *Peano's Conception of a Single Infinite Cardinality*, «HOPOS. The Journal of the International Society for the History of Philosophy of Science», n. 13-2, 2023, pp. 241-260.

[24] G. Peano, *Arithmetices principia, nova methodo exposita*, Bocca, Torino 1889.

[25] Id., *Sul concetto di numero*, in «Rivista di matematica», n. 1, 1891, pp. 87-102.

[26] «Con *num a* intenderemo "il numero degli individui della classe *a*". [...]. Il segno *num* è un segno d'operazione che ad ogni classe fa corrispondere o un *N*, o lo 0, o l'∞», (*ibid.*, p. 101).

is just the (finite) number of its elements, i.e., a natural number. But then, Peano states that *num a* is not always a natural number, since the set of natural numbers does not include "zero" and "infinity"[27].

This is, arguably, the first appearance of the *infinite* in Peano's work as "cardinality" of a non-finite set (a class, in his jargon). That Peano thought that "∞" was a quantity as any other finite quantity can be deduced from the fact that Propositions 3 and 4 of the article's section 9, taken together, extend the arithmetical operation of addition to "∞". First, he defines addition on *all* quantities: «3. If *a* and *b* are two non-empty and finite classes having no element in common, then the number of elements of the set of the two classes *a* and *b* is equal to the sum of the number of *a*s and *b*s»[28]. Then he notes that the proposition holds even if one of the two classes, or even both, contain *infinite* elements; but now, he says, we have that:

$$x + \infty = \infty + x = \infty$$

where x is a finite quantity, and

$$\infty + \infty = \infty$$

As far as the behaviour of "∞" with respect to its "parts" is concerned, in the next proposition, Peano notes: «4. If the classes *a* and *b* are such that the second is contained in the first, and the class *b* is non-empty, and is not equal to *a*, and if the number of *a*s is finite, then the number of *b*s is also finite, and is less than the number of *a*s». Finally, he observes that «this proposition ceases to be valid if *num a* = ∞»[29].

So, the examination of *Sul concetto di numero* reveals that Peano had a conception of the mathematical infinite whereby:

1. There exists just *one* infinite

27 «Data una classe *a* non sempre *num a* è un *N*, poiché *N* non comprende né lo zero, né l'infinito», (*ibid.*, p. 101).

28 «Essendo *a* e *b* due classi non nulle e finite, non aventi alcun individuo comune, allora il numero degli individui appartenenti all'insieme delle due classi *a* e *b* vale la somma dei numeri degli *a* e dei *b*» (*ibid.*, p. 101).

29 «Questa proposizione cessa di esistere se *num a* = ∞» (*ibidem*).

2. This means, among other things, that all non-finite sets have the same infinite cardinality, that is, '∞';
3. More complex infinitary quantities based on '∞' may be taken to be, to all intents and purposes, the *same* quantity

This is what I have called Peano's "single-cardinality" conception of the infinite[30]. In subsequent works, in which the concept of the infinite is re-examined (in particular in the many volumes of the *Formulario di matematica*)[31], the conception is first restated (with some minor variations) but then finally abandoned in favour of the set-theoretic conception. One of the reasons why Peano may have eventually adopted the set-theoretic conception may just have been that he became more familiar with set theory thanks to his direct exchanges with Cantor in late 1890s[32]. In any case, the chronology of these works helps us state, with a high degree of accuracy, that at the time of the composition of the inconsistency proof, Peano still supported the "single-cardinality" conception.

As a consequence of this, in particular, of property 3) above, Peano may have then thought that Cantor's transfinite ordinals, although clearly differing from one another *qua* ordinals, may be taken to be *equal*, to be, that is, the same *infinite*. By viewing the "single-cardinality" conception of the infinite as the main reason Peano had for assimilating Cantor's transfinite ordinals to his own numbers (and segments) in his inconsistency proof, one does need to conjecture, as Freguglia does, that Peano mistook Cantor's transfinite ordinals for $\aleph_0$.

My interpretation is also consistent with and, in fact, corroborated by, Veronese's and Vivanti's comments. As seen, the two thought that Peano knew set theory well enough to be able to distinguish ordinals and cardinals, and one ordinal from another. However, they also held, or, at least Veronese did, that, in Peano's view, such geometric

[30] A brief examination of Peano's conception may also be found in P. Mancosu, *Abstraction and Infinity*, Oxford University Press, Oxford 2016.

[31] G. Peano, *Formulario di matematica*. Vol. 1, Bocca, Torino 1895, Vol. 2, Sez. 1, Bocca, Torino 1897, Vol. 2, Sez. 2, Bocca, Torino 1898, Vol. 2, Sez. 3, Bocca, Torino 1899, Vol. 3, Carré et Naud, Paris 1901, Vol. 4, Bocca, Torino 1903, Vol. 5, Bocca. Torino 1908.

[32] Cf. C. Ternullo, I. Fascitiello, *Peano's Conception of a Single Infinite Cardinality*, cit., section 3.2.

quantities (and segments) as ∞u were "unlimited", that is, they were manifestations of what Peano viewed as a unique infinite cardinality that could not be *transcended*.

5. *Summary*

Peano's proof aims to show that actual (geometric) infinitesimals are bound to be inconsistent. The crux of the proof consists in showing that they behave in a fairly different manner than finite geometric quantities (finite bounded segments). These can be added and multiplied, and are liable to become *greater* and *smaller*; geometric infinitesimals, on the contrary, even when added or multiplied, always remain the same.

In the paper, I have conjectured that the proof's main driving force is a conception of the infinite that I have called "single-cardinality" conception that Peano did not see as inconsistent with Cantor's.

Early critics of Peano's proof, such as Vivanti and Veronese, and more recently Freguglia, have brought to the fore the questionable, and inaccurate, use of set theory in Peano's proof and tried to remedy this by offering alternative interpretations based, again, on set theory. My own interpretation, on the contrary, lays emphasis on the originality of Peano's synthesis of his and Cantor's approach, which ultimately led him to produce a proof of the purported inconsistency of infinitesimals.

The value of the proof should, of course, be measured against the strengths and the shortcomings in the mathematical apparatus. However, even if the proof were formally impeccable, it would still be debatable that it could really be able to show that infinitesimals are inconsistent. But, as said, this is a matter that lies beyond the scope of this article.

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