What Is Set-Theoretic Truth?

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Philosophy of mathematics has intersections with many other philosophical disciplines: logic, metaphysics, epistemology, and, of course, with several branches of mathematics itself.

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- **What is set-theoretic truth?**

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Set theory helps fully vindicate the actualist point of view.

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Finally, for a set A, one defines its power-set $\wp(A)$, i.e., the set of all subsets of A, and produce the new numbers: $\beth_0 = \aleph_0$, $\beth_1 = 2^{\aleph_0}$, $\beth_2 = 2^{2^{\aleph_0}}$, ...

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With AC finally onboard, Zermelo could finally vindicate Cantor's *set-theoretic actualism*.

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- (Extensionality) Sets with the *same* elements are *equal*.

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NB. Through the ZFC axioms, one can reproduce (in a sense, also motivate) the whole of (Cantorian) set theory and avoid the *paradoxes*. However, intuitively one may also motivate the axioms in a different way: through the *iterative concept of set*: sets 'come' in (well-founded) *stages* (see [Boolos, 1971], [Potter, 2004]).

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$V \models ZFC$ (but note that this can't be stated as a theorem in ZFC, if ZFC is consistent).

V/cont'd.

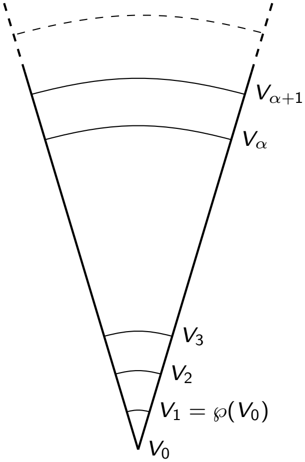

Figure: The Cumulative Hierarchy

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Σ^1_2-Sets of Reals Measurability Hypothesis

Σ^1_2-sets of reals are Lebesgue measurable (the Lebesgue measure of a set A is a number $\mu \in [0, \infty]$).

Independence Proofs

In order to show that ϕ is *formally undecidable* in ZFC (for that matter, any sufficiently strong theory **T**), one proceeds as follows:

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Two fundamental *models* are: L and $V[G]$ (for some G), respectively, an *inner* and *outer* model of V.

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Models

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Each of these settles any of the *independent* statements above in *different* (sometimes *incompatible*) ways.

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- (1) What is 'intuition of V'?
- (2) There are just too many ZFC+A's, too. How should one *select* them? (In any case, indeterminacy will still badly affect any ZFC+A.)
- (3) Categoricity arguments do not prove that there is a single universe, only that a theory of a single universe may be produced through *strengthening* the logical framework.

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Problems: (1) pathological universes, such as those satisfying ZFC+¬Con(ZFC) must exist; (2) this is a form of (self-undermining) skepticism; (3) it isn't formalisable.

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Problems: there are many other 'outer' models of V which are not included in this multiverse concept.

Set-Generic Multiverse/Cont'd.

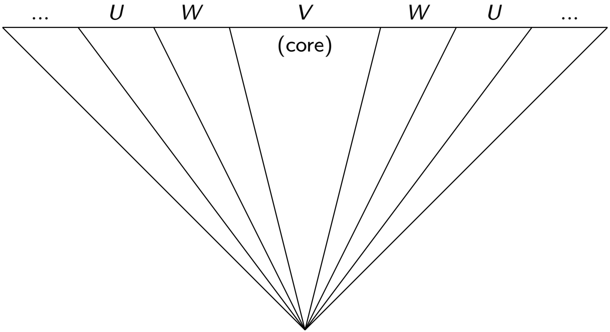

Figure: The Set-Generic Multiverse with a Core

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Problem: the procedure will just yield a *syntactic* multiverse, as there is no *semantics* for V-logic, if V is *uncountable* (but fixes may be found).

The V-logic multiverse/Cont'd

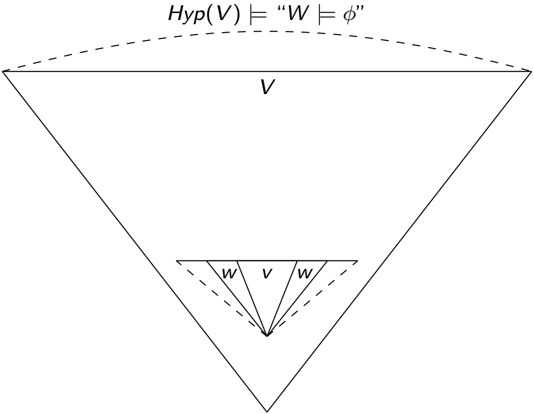

Figure: Outer Models in the V-logic Multiverse

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- Insofar as set theory is the *foundation* of mathematics, and mathematics just needs 'some' (very little) set theory, then we should not care too much about the notion of truth for the 'whole of set theory'
- (Finitist's Revenge) Set theory is (globally) *meaningless*

The End

Discussion. (ah, thanks for the attention!)

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