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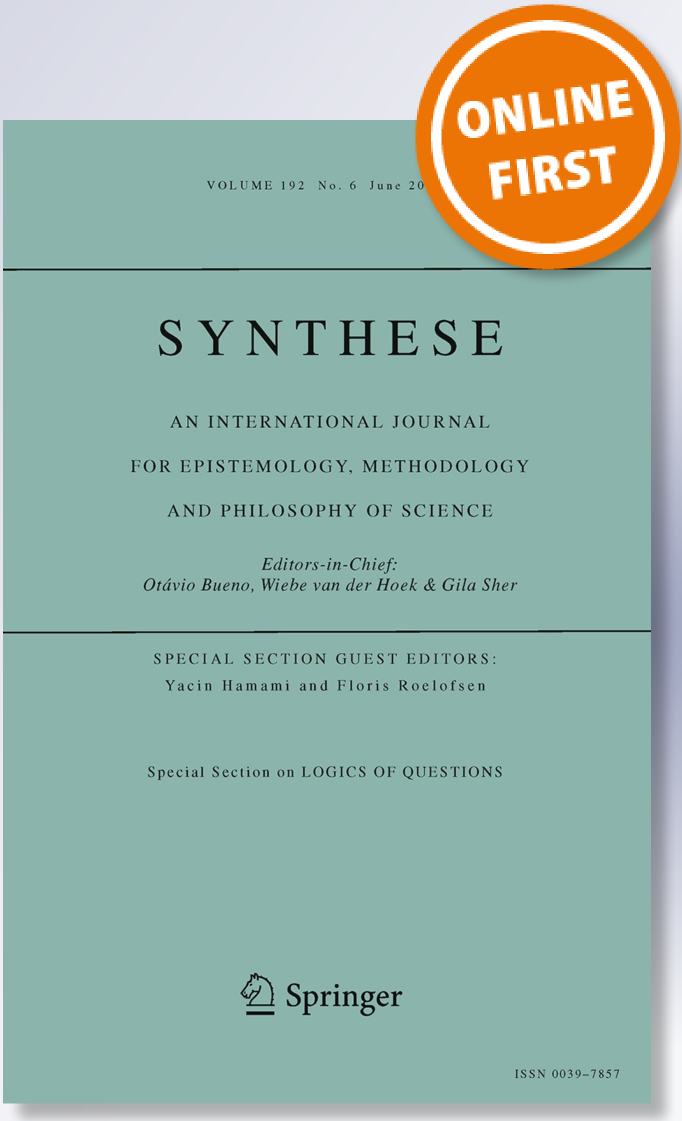

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Multiverse conceptions in set theory

Carolin Antos[1] · **Sy-David Friedman**[1] ·
Radek Honzik[1,2] · **Claudio Ternullo**[1]

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Abstract We review different conceptions of the set-theoretic multiverse and evaluate their features and strengths. In Sect. 1, we set the stage by briefly discussing the opposition between the 'universe view' and the 'multiverse view'. Furthermore, we propose to classify multiverse conceptions in terms of their adherence to some form of mathematical realism. In Sect. 2, we use this classification to review four major conceptions. Finally, in Sect. 3, we focus on the distinction between actualism and potentialism with regard to the universe of sets, then we discuss the Zermelian view, featuring a 'vertical' multiverse, and give special attention to this multiverse conception in light of the hyperuniverse programme introduced in Arrigoni and Friedman (Bull Symb Logic 19(1):77–96, 2013). We argue that the distinctive feature of the multiverse conception chosen for the hyperuniverse programme is its utility for finding new candidates for axioms of set theory.

Keywords Set theory · Universe of sets · Set-theoretic multiverse · Hyperuniverse programme · New axioms of set theory

1 The set-theoretic multiverse

1.1 Introduction

Recently, a debate concerning the set-theoretic multiverse has emerged within the philosophy of set theory, and it is plausible to expect it to remain at centre stage for a long time to come. The 'multiverse' concept was originally triggered by the

✉ Claudio Ternullo
claudio.ternullo@univie.ac.at

1 KGRC, Vienna, Austria

2 Charles University, Prague, Czech Republic

independence phenomenon in set theory, whereby set-theoretic statements such as CH (and many others) can be shown to be independent from the ZFC axioms by using different *models* (universes). The collection of *some* or *all* of these models constitutes the set-theoretic multiverse.

While the existence of a set-theoretic multiverse is a well-known mathematical fact, it is far from clear how one should properly conceive of it, and in what sense the 'multiverse phenomenon' bears on our experience and conceptions of sets. Furthermore, as is frequent in the philosophy of mathematics, the issue of the nature of the multiverse intersects other no less prominent issues, concerning the nature of mathematical objectivity, ontology and truth. For instance, in what sense may the set-theoretic multiverse imply a revision of our 'standard' conception of truth? What are universes within the multiverse like and how should they be selected? Is the multiverse a merely transient phenomenon or can it legitimately claim to represent the ultimate set-theoretic ontology? Is there still any chance for the 'universe view' to prevail notwithstanding the existence of the multiverse? These are only a few examples of the philosophical issues one may want to examine and, in what follows, some of these questions will be addressed. Our goal is twofold: to provide an account of the positions at hand in a systematic way, and to present our own theory of the multiverse.

In order to fulfill the first goal, in Sect. 2, we will give an overview of some of the available conceptions, whereas in Sect. 3 we will introduce one further conception, which befits the goals of the hyperuniverse programme. Our focus will be more on philosophical features than on mathematical details, although, sometimes, a more accurate mathematical account will inevitably have to be given.

1.2 The 'universe view' and the 'multiverse view'

Let us preliminarily explain, in general terms, what a 'multiverse view' amounts to. In particular, we clarify how it can and should be contrasted to a 'universe view'.

The universe of sets, V, is the cumulative hierarchy of all sets, starting with the empty set and iterating, along the ordinals, the power-set operation at successor stages and the union operation at limit stages. The ZFC axioms will be our reference axioms and we know that these axioms are unable to specify many relevant properties of the universe. For instance, the axioms do not tell us what the size of the continuum is or whether there exist measurable cardinals (provided they are consistent with ZFC). In fact, different versions of V, obtained through model-theoretic constructions, are compatible with the axioms. Set-theorists work with lots of these constructions: set-generic or class-generic extensions (obtained through forcing), inner models, models built using, for instance, ultrafilters, ultraproducts, elementary embeddings and so on. Some of these are sets, others are classes, some satisfy CH and some do not, some satisfy $\diamondsuit$ and some do not, and so on. A huge variety of combinatorial possibilities comes with the study of set-theoretic models and the bulk of contemporary set theory consists in studying, classifying and producing models not only of ZFC, but also of some of its extensions.

Two immediate questions may arise: how should we interpret such a situation in relation to the uncritical assumption that V denotes a 'fixed' entity, a determinate object? What is the relationship between the universe and the multiverse? The two possible responses to such questions yield what we may see as the two basic possible philosophical alternatives at hand: the universe view and the multiverse view.

The former conception is characterised as follows: there is a definite, unique, 'ultimate' set-theoretic structure which captures all true properties of sets. Its supporters are aware that the first-order axioms of set theory are satisfied by different structures, but from this fact they only infer that the currently known axioms are not sufficient to describe *the* universe in full. They may think that set-theoretic indeterminacy will be significantly reduced by adding new axioms which will provide us with a more determinate picture of the universe, but they may also believe that we will never reach a complete understanding of the universe, after all.

On the other hand, the multiverse view can be characterised in the following way: there is a wide realm of models which satisfy the axioms, all of which contain relevant, sometimes alternative, pieces of information about sets. Each of these, or, at least, some of these represent all equally legitimate universes of sets and, accordingly, there is no unique universe, nor should there be one. The multiverse view supporter believes that the absence of a unique reference of the set-theoretic axioms will not and cannot be repaired: set theory is about different realms of sets, each endowed with properties which differentiate it from another.

In light of current set-theoretic practice, both conceptions are legitimate and tenable, and both are problematic. It is easy to see why. In very rough terms, the universe view supporter owes us an account of how, notwithstanding the existence of one single conceptual framework, we can think in a perfectly coherent way of different alternative frameworks. If she thinks that such alternative frameworks are not definitive, then she has to explicate why they are epistemically reliable (that is, why they give us 'true' knowledge about sets). The multiverse view supporter, on the other hand, owes us an account of how, notwithstanding the existence of multiple frameworks, one can always imagine each of them as being 'couched' within V. Granted, such frameworks may well be *mutually* incompatible, but, surely, they must *all* be compatible with V.

1.3 A proposed classification

There are a lot more nuanced versions of each of the two positions. In order to address more closely what we believe are the most relevant, we want to propose a systematic way to group them. We will add one further criterion of differentiation, that of their commitment to some form of *realism*. In plain terms, commitment to realism measures how strongly each conception holds that the universe or the multiverse exist *objectively*.

It is fairly customary in the contemporary philosophy of mathematics to differentiate realists in ontology from realists in truth-value, and we will pre-eminently focus here on *realism in ontology*.[1] Thus, the universe view or multiverse view may split into two further positions, according to whether one is a realist or a non-realist universe view or multiverse view supporter. Such a differentiation yields, in the end, four positions,

[1] For the distinction and its conceptual relevance within the philosophy of mathematics, see Shapiro's introduction to Shapiro (2005) or Shapiro (2000).

each of which, we believe, has had some tradition in the philosophy of mathematics or has been influential in, or relevant to the current debate in the foundations of set theory.

A realist universe view may alternatively be described as that of a Gödelian platonist. Although Gödel's views may have changed during his lifetime, it seems rather plausible to construe his many references to the reality of sets in the context of a realist (platonist) universe view.[2]

But one could be a universe view supporter without believing in the external existence of the universe. One may, for instance, view the universe view as only 'practically' confirmed on the basis of some specific mathematical results. For instance, Maddy's 'thin realist' would presumably hold that the universe view is preferable as long as it better fits set theory's first and foremost purpose of producing a 'unified' arena wherein all mathematics can be carried out.[3]

The realist multiverse view supporter fosters a peculiar strain of realism, based on the assumption that there are different, alternative, 'platonistically' existing concepts of sets instantiated by different, alternative universes or, alternatively, that there are different universes, which correspond to alternative concepts of set. As known, such a conception has been set forth and articulated in full as a new version of platonism known as *full-blooded platonism* (FBP).[4] Within the context of the debate we are interested in, this conception has been recently advocated by Hamkins, and we will devote substantial efforts to examining its features.

Finally, the non-realist multiverse view supporter is someone who does not believe in the existence of universes and, in particular, does not believe in the existence of a single universe. To someone with these inclinations, the multiverse is a 'practical' phenomenon, so to speak, with which one should deal as with any other fact of mathematical practice. This position seems to represent the basic 'uncommitted' viewpoint, but may be further elaborated using, for instance, the formalist viewpoint. A typical representative of this attitude is Shelah, whose position we will describe in Sect. 2.

As we said above, one may be a realist in truth-value and a non-realist in ontology and vice versa, one may be both and one may be neither. The introduction of one further criterion of differentiation, namely the commitment to realism in truth-value, in principle, would give us further available positions, but the four conceptions briefly

[2] See, for instance, the following oft-quoted passage in his Cantor paper: "It is to be noted, however, that on the basis of the point of view here adopted [that is, the 'platonistic conception', *our note*], a proof of the undecidability of Cantor's conjecture from the accepted axioms of set theory (in contradistinction, e.g., to the proof of the transcendency of π) would by no means solve the problem. For if the meaning of the primitive terms of set theory as explained on page 262 and in footnote 14 are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality in which Cantor's conjecture must be either true or false." (Gödel (1947), in Gödel (1990), p. 260). For an account of the development of Gödel's conceptions, see, for instance, Wang's books, Wang (1974, 1996) and also van Atten and Kennedy (2003).

[3] For the full characterisation of Maddy's 'thin' realist, see Maddy (1997, 2011). A possible middle ground between a realist and a non-realist universe view has been described by Putnam in his (1979). A 'moderate realist', as featured there, would be someone who does not buy into full-blown platonism but who, at the same time, still believes to be able to find evidence in favour of some sort of 'ultimate' universe.

[4] See, especially, Balaguer (1995, 1998).

described above are sufficient to cover the whole spectrum of the existing conceptions, at least for now.[5]

2 Multiverse conceptions

We will now proceed to examine multiverse conceptions in detail. We will, in turn, present four positions, before introducing ours. As anticipated, Hamkins and Shelah will be our featured representatives of, respectively, a realist and a non-realist multiverse view. We will also be scrutinising two further positions, whose authors, it seems to us, are, in fact, universe view supporters, and these are Woodin's and Steel's. Incidentally, this fact testifies to the essential non-rigidity of set-theorists' stances in the practical arena: the universe view and the multiverse view are variously advocated or rejected philosophically, but the universe and the multiverse constructs are used indifferently by set-theorists as tools to study sets and determine their properties. The last two authors have presented their own version of the multiverse either to subsequently discard it (Woodin) or to suggest ways to reduce it to a universe view (Steel) and, through examining them, we also hope to receive some insight on how and why one may, at some point, get rid of the multiverse.

2.1 The realist multiverse view

2.1.1 The general framework

Hamkins has made a full case for a realist multiverse view in his Hamkins (2012). Before examining it in detail, let us briefly summarise it. Set theory deals with different model-theoretic constructions, wherein the truth-value of relevant set-theoretic statements may vary. The only way to make sense of this phenomenon is to acknowledge that there are different *set concepts*, each of which is instantiated by a specific model-theoretic construct, that is, a universe of sets. There is no *a priori* reason to ban any model-theoretic construction from the multiverse: any universe of sets is a legitimate member of the set-theoretic multiverse. This means that even such controversial models as ill-founded models are granted full citizenship in the multiverse.[6]

[5] Incidentally, it is not clear whether a realist in truth-value is best accommodated to the universe view. For instance, take Hauser, who seems to be only a realist in truth-value. He says: "At the outset mathematical propositions are treated as having determinate truth values, but no attempt is made to describe their truth by relying on a specific picture of mathematical objects. Instead one seeks to exhibit the truth or falsity of mathematical propositions by rational and reliable methods" (Hauser 2002, p. 266). From this, it is far from clear that one single picture of sets would have to be found anyway, if not at the outset, at least in due course, presumably after the truth-value of such statements as CH has been reliably fixed. Hauser has also addressed truth-value realism and its conceptual emphasis on *objectivity* rather than on *objects* in Hauser (2001). See also Martin (1998).

[6] Hamkins epitomises this conception through the adoption of the naturalistic maxim 'maximise', by virtue of which one should not place "undue limitations on what universes might exist in the multiverse. This is simply a higher-order analogue of the same motivation underlying set-theorists' ever more expansive vision of set theory. We want to imagine the multiverse as big as possible" (Hamkins 2012, p. 437).

Furthermore, any universe in the multiverse describes an *existing* reality of sets. The latter thesis implies that any model-theoretic construct should also be taken to describe an existing reality in the platonistic sense.

This peculiar form of multiverse realism is the crux of Hamkins' conception and, thus, needs an extended commentary.

First of all, Hamkins leaves no doubt as to the platonistic character of his conception:

> The multiverse view is one of higher-order realism—Platonism about universes—and I defend it as a realist position asserting actual existence of the alternative set-theoretic universes into which our mathematical tools have allowed us to glimpse. (Hamkins 2012, p. 417)

Now, as anticipated in Sect. 1, it may not have escaped other commentators that the conception Hamkins formulates in the passage above is connected to that peculiar version of platonism, due to Balaguer, known as full-blooded platonism (FBP). FBP is, ontologically, a richer version of platonism, as, on this conception, *any* theory of sets describes an existing realm of objects (that is, a universe of sets), and, consequently, FBP has to accommodate the platonistic conception of truth to this enlarged ontological realm. In Balaguer's words:

> It is worth noting that what FBP does *not* advocate is a *shift* in our conception of mathematical truth. Now, it *does* imply (when coupled with a corresponding theory of truth) that the consistency of a mathematical sentence is sufficient for its truth. [...] What mathematicians *ordinarily* mean when they say that some set-theoretic claim is true is that it is true of the *actual* universe of sets. Now, as we have seen, according to FBP, there is no *one* universe of sets. There are many, but nonetheless, a set-theoretic claim is true just in case it is true of *actual* sets. What FBP says is that there are so many different kinds of sets that every consistent theory is true of an *actual* universe of sets. (Balaguer 1995, p. 315)

Balaguer gives one further neat exemplification of the situation described above, by explaining that:

> According to FBP, both ZFC and ZF+ not-C[7] truly describe parts of the mathematical realm; but there is nothing wrong with this, because they describe *different* parts of that realm. This might be expressed by saying that ZFC describes the universe of sets_1, while ZF+not-C describes sets_2, where sets_1 and sets_2 are different kinds of things. (Balaguer 1995, p. 315)

Analogously, Hamkins sees universes of sets as being tightly related to specific *set concepts*, the latter being, presumably, embodied by an axiom or a collection of axioms (in Balaguer's example, sets_1 is the universe or region of the multiverse which instantiates the set concept expressed by AC and sets_2 that which instantiates the set concept expressed by the negation of AC):

[7] ZF+ the negation of the Axiom of Choice.

> Often, the clearest way to refer to a set concept is to describe the universe of sets in which it is instantiated, and in this article I shall simply identify a set concept with the model of set theory to which it gives rise. (Hamkins 2012, p. 417)

It is not entirely clear from what Hamkins says whether a set concept should be automatically and uniquely identified with the universe(s) that instantiate it, or whether concepts of sets have an independent (and *prioritary*) status, something which would presumably differentiate Hamkins' position from Balaguer's. What is sure is that the correspondence between set concepts and instantiating universes should be construed in terms of a correspondence between axioms and models, as demonstrated by the following general observation:

> The background idea of the multiverse, of course, is that there should be a large collection of universes, each a model of (some kind of) set theory. There seems to be no reason to restrict inclusion only to ZFC models, as we can include models of weaker theories ZF, ZF^-, KP and so on, perhaps even down to second order number theory, as this is set-theoretic in a sense. (Hamkins 2012, p. 436)

So much for the ontology of the multiverse. As we have seen, its underlying philosophy does not seem to differ to a significant extent from that of Balaguer's FBP-ist. Where, on the contrary, Hamkins seems to supplement it, is on set-theoretic truth, and, in particular, with regard to the truth-value of the undecidable statements. An FBP-ist is supposed to be very liberal on this: the answer to a set-theoretic problem (and, possibly, also to a non-set-theoretic problem which has a strong dependency upon set theory) depends on the universe of sets one is talking about. Accordingly, CH may be true in some universes and false in others, but there is no *a priori* reason to consider one of the answers provided by a universe in the multiverse as more relevant or more strongly motivated than any other.

On this point, Hamkins seems to want to expand on FBP. Although, at the purely ontological level, all universes are equally legitimate, set-theoretic practice may still dictate which are more relevant in view of specific needs. We should still pay attention to specific versions of truth within universes, as presumably there are some which look more 'attractive' than others, as explained in the following quote:

> ...there is no reason to consider all universes in the multiverse equally, and we may simply be more interested in the parts of the multiverse consisting of universes satisfying very strong theories, such as ZFC plus large cardinals. The point is that there is little need to draw sharp boundaries as to what counts as a set-theoretic universe, and we may easily regard some universes as more set-theoretic than others. (Hamkins 2012, pp. 436–437)

Elsewhere, he expresses such concerns even more neatly. For instance, when he talks about CH, he says that

> On the multiverse view, consequently, the continuum hypothesis is a settled question; it is *incorrect* [*our italics*] to describe the CH as an open problem. The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extent to which it holds and fails in the multiverse, about how to achieve it or its negation in combination with other diverse set-theoretic

> properties. Of course, there are and will always remain questions about whether one can achieve CH or its negation with this or that hypothesis, but the point is that the most important and essential facts about CH are deeply understood, and these facts constitute the answer to the CH question. (Hamkins 2012, p. 429)

The emphasis, here, is more on the fact that a shared solution to the Continuum Problem is represented by our 'knowledge' of how CH varies across the multiverse, rather than on its truth-value in specific universes. As a consequence, here Hamkins seems to conjure an epistemically 'active' role for his multiverse conception, as a study of the *relationships* among universes which may provide us with detailed knowledge of the alternative answers to mathematical problems. Such a view is also expressed in the quote below:

> On the multiverse view, set theory remains a foundation for the classical mathematical enterprise. The difference is that when a mathematical issue is revealed to have a set-theoretic dependence, then the multiverse is a careful explanation that the mathematical fact of the matter depends on which concept of set is used, and this is almost always a very interesting situation, in which one may weigh the desirability of various set-theoretic hypotheses with their mathematical consequences. (Hamkins 2012, p. 419)

2.1.2 Problems with the realist multiverse view

We now want to proceed to examine some potential difficulties with the realist multiverse view. In fact, in what follows, these are formulated as objections to FBP rather than to Hamkins' views. However, if one believes, as we do, that Hamkins' views are modelled upon (or, at least, connected to) the former, then the realist multiverse view supporter should take such objections to FBP very seriously.

One issue is that of whether we have sufficient grounds to assert that the existence of different models can be construed in terms of the existence of different universes of sets instantiating different concepts (and theories) of sets, as required by FBP (and presumably, as we have seen, also by the Hamkinsian multiverse view supporter).

A second, but parallel, issue is that of whether models (universes) are sufficiently characterised, conceptually, to be viewed as more than mere alternative characterisations of a unique existing universe, V. This is precisely the issue we mentioned at the beginning: in fact, each model can be seen to be living 'inside' V. This issue inevitably puts some pressure on the realist multiverse view supporter, who believes that there is no 'real' V and, most crucially, that knowledge about set-theoretic truth depends on knowledge of universes other than V.

As far as the first issue is concerned, first we wish to review an objection to FBP by Colyvan and Zalta. In Colyvan and Zalta (1999), the authors argue that, by FBP, when we consider two different models M and N, we should think of them as two entirely different realms of sets. But this is hardly the case. For instance, all forcing extensions of a transitive model of the axioms of ZFC leave the truth-value of sentences at the arithmetical level unchanged, and, thus, it is hard to imagine that the finite numbers in M are different from the finite numbers in N.

If, on the other hand, M and N contain the same finite numbers, then these models will hardly be *entirely* different realms of sets. For instance, $\mathcal{P}(\omega)$ in M may be different from $\mathcal{P}(\omega)$ in N, but a set like 7 will be the same in both.

It is not clear whether Hamkins is aware of this criticism, when, at some point in his paper, he even challenges the fixedness of the concept of natural number. He argues that Zermelo-style categoricity arguments, which would give us grounds to believe the universe view, are unpersuasively based on an absolute concept of set. This may extend to categoricity arguments with regard to arithmetic, insofar as

> ...although it may seem that saying "1, 2, 3, ... and so on" has to do with a highly absolute concept of finite number, the fact that the structure of the finite numbers is uniquely determined depends on our much murkier understanding of which subsets of the natural numbers exist. [...] My long-term expectation is that technical developments will eventually arise that provide a forcing analogue for arithmetic.. (Hamkins 2012, p. 14)

We do not know on what grounds Hamkins makes such a prediction and whether the discovery of a forcing analogue for arithmetic would help him rebut the mentioned objection efficaciously. In any case, if we take FBP as asserting that, whenever we have two different models, then we have entirely different sets of objects in them, we inevitably fall back on the thesis that there are as many types of finite sets as models, something which seems to defeat our well-established, pre-theoretic assumption of the full determinacy of finite numbers. If, on the contrary, universes may share some, but not all sets, then it is less easy to recognise them as entirely alternative realms of sets (universes), although the latter case is less problematic.

To introduce the second issue, we will start with a quote from Potter's (2004). The author says:

> ...for a view to count as realist, [...], it must hold the truth of the sentences in question to be metaphysically constrained by their subject matter more substantially than Balaguer can allow. A realist conception of a domain is something we win through to when we have gained an understanding of the nature of the objects the domain contains and the relations that hold between them. For the view that bare consistency entails existence to count as realist, therefore, it would be necessary for us to have a quite general conception of the whole of logical space as a domain populated by objects. But it seems quite clear to me that we simply have no such conception. (Potter 2004, p. 11)

Potter's criticism goes to the heart of the FBP-ist's conception, which essentially consists in the claim that 'consistency guarantees existence'. That this doctrine is tenable and conceptually cogent is crucial to Hamkins' purposes: if members of the multiverse do not exist on the grounds of consistency alone, then the entire multiverse construct might be seen as shaky. It should be clarified that the problem, here, is not whether the form of existence of mathematical entities set forth by FBP-ists is inadequate to hold that FBP is a realist conception, but rather that the form of existence set forth by FBP-ists is not sufficient to spell out a multiverse conception.

The crucial case study is the 'ontology of forcing'. The realist multiverse view supporter commits himself to asserting that the forcing extension of the universe V

associated to a V-generic filter G, $V[G]$, really exists.[8] But how can he do that, if V is everything there is? The response is contained in the following remark:

> On the multiverse view, the use of the symbol V to mean "the universe" is something like an introduced constant that might refer to any of the universes in the multiverse, and for each of these the corresponding forcing extensions $V[G]$ are fully real. (Hamkins 2012, p. 5)

Once he has ascertained that there is a consistent reading of such objects as $V[G]$, via FBP he can legitimately say that such objects exist. But that is precisely where we find Potter's objection strike at the roots. What kind of existence is that? Suppose Potter is right, and set-generic extensions cannot be said to be existing only by virtue of their consistency. That would put the multiverse view supporter in trouble, as he might face up with the objection that set-generic extensions are, in fact, illusory. This is precisely the strategy used by a universe view supporter. Hamkins acknowledges this fact himself:

> Of course, one might on the universe view simply use the naturalist account of forcing as the means to explain the illusion: the forcing extensions don't really exist, but the naturalist account merely makes it seem as though they do. (Hamkins 2012, p. 10)

Yet, Hamkins thinks that FBP is more in line with our experience of these objects as expressed within the naturalist account:

> ...the philosophical position [higher-order realism, *our note*] makes sense of our experience—in a way that the universe view does not—simply by filling in the gaps, by positing as a philosophical claim the actual existence of the generic objects which forcing comes close to grasping, without actually grasping. (Hamkins 2012, p. 11)

However, this position is, at least, controversial. Admittedly, forcing comes close to grasping generic objects, but actually never grasps them. How may all this ever help us believe that these objects are existent? Is consistency alone a sufficient reason?

Similar concerns have been expressed by Koellner, in his Koellner (2013), where the author extends them to other model-theoretic constructions:

> In summary, on the face of it, all three methods provide us with models that are either sets in V or inner models (possibly non-standard) of V or class models that are not two-valued. In each case one sees by construction that (just as in the case of arithmetic) the model is non-standard. One can by an act of imagination treat the new model as the "real" universe. The broad multiverse position is a *consistent* position. But we have been given no reason for taking that imaginative leap. (Koellner 2013, p. 22)

[8] Hamkins defines this interpretation of forcing 'naturalist', as opposed to the 'original' interpretation of forcing, whereby one starts the construction with a countable transitive ground model M and extends it to an $M[G]$, by adding an M-generic filter G.

In fact, as we have seen, the reason we have been given for the 'imaginative leap' Koellner is referring to here, is the central ontological position of FBP, that 'consistency guarantees existence'. But if we doubt that such a position is at all tenable, then it is reasonable to assert that the existence of a multiverse, at least in the terms conjured by Hamkins, may be seen as not entirely unproblematic.

2.2 The non-realist multiverse view

None of these difficulties has to be addressed by a non-realist multiverse view supporter. This kind of *pluralist* does not believe in the existence of universes and, thus, is likely to dismiss all the ontological concerns we have previously reviewed. This person may see the set-theoretic multiverse not as a structured, or independent, for that matter, reality, but only as a phenomenon arising in practice. As a consequence, he may not attach any relevance either to 'real V' or to the 'multiverse': in this person's view, these are just labels one may use indifferently on the basis of one's personal needs and theoretical convenience. Given such presuppositions, this person may see the multiverse essentially as a tool to produce independence proofs.

As Koellner has painstakingly shown in one of his recent articles, one of the most committed proponents of this form of 'radical pluralism' has been Carnap.[9] The Carnapian pluralist typically believes that any theory, say ZFC+CH or ZFC+¬CH, has its own appeal and that adopting either is only a matter of *expedience*. Given that the meaning of the axioms is not dependent upon any prior knowledge of their content, the only theoretical concern a Carnapian pluralist has is that of finding what one can prove from those axioms. It follows that models are only needed indirectly, to understand which statements are provable from the axioms and, thus, to allow us carry out independence proofs.

The kind of realist we have scrutinised in the preceding section may also be viewed as a radical pluralist as far as truth is concerned. However, as we have seen, the Hamkinsian multiverse view supporter has a somewhat different view of truth, insofar as she may acknowledge that it is not only theoretical expedience which dictates our choices of the axioms and of the corresponding models.

The typical attempt to counteract pluralism consists in showing that (1) *expedience* is not sufficient to adopt a theory[10]; and/or (2) that issues of meaning cannot be entirely circumvented. An example of the latter strategy is given by Gödel's response to Errera's attacks in his 1964 version of his Cantor paper. Errera had stated that set theory is bifurcated by CH in the same way as geometry by Euclid's fifth postulate, and that one should feel free to see each of the two theories as equally justified. Gödel attempted to rebut this view, by pointing out that even the axioms of Euclidean geometry may have a fixed meaning: at least they refer to laws concerning bodies and, accordingly, a decision on their truth might depend on that given system of physical objects.

Within the community of set-theorists, it is Shelah who has voiced most vividly the non-realist multiverse conception. In his 'foundational' paper, he says:

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[9] See Koellner (2009b), especially Sect. 2.

[10] This is, for instance, Koellner's line of attack in the aforementioned paper.

> My mental picture is that we have many possible set theories, all conforming to ZFC. I do not feel "a universe of ZFC" is like "the Sun", it is rather like "a human being" or "a human being of some fixed nationality". (Shelah 2003, p. 211)

Shelah is, admittedly, a 'mild' formalist,[11] but he does not seem to adhere to the full-blown radical pluralistic point of view about truth. Instead, he talks about different degrees of 'typicality' of models of the axioms (in this case, ZFC). Always referring to models of ZFC as 'citizens', he specifies:

> ...a typical citizen will not satisfy $(\forall\alpha)[2^{\aleph_\alpha} = \aleph_{\alpha+\alpha+7}]$ but will probably satisfy $(\exists\alpha)[2^{\aleph_\alpha} = \aleph_{\alpha+\alpha+7}]$. However, some statements do not seem to me clearly classified as typical or atypical. You may think "Does CH, i.e., $2^{\aleph_0} = \aleph_1$ hold?" is like "Can a typical American be Catholic?" (Shelah 2003, p. 211)

In fact, it should be pointed out that radical, unmitigated pluralism is hardly the preferred choice among pluralists, as all of them seem to be keen on finding correctives (such as the aforementioned preference for 'typicality'). For instance, Field says:

> ...we can still advance aesthetic criteria for preferring certain values of the continuum over others; we must now view these not as *evidence that* the continuum has a certain value, but rather as *reason for refining our concepts so as to give* the continuum that value, [...]. (in Dales and Oliveri 1998, p. 300)

Analogously, Balaguer finds that:

> There are at least two ways in which the FBP-ist can salvage the objective bite of mathematical disputes. The first has to do with the notion of *inclusiveness* or *broadness*: the dispute over CH, for instance, might be construed as a dispute about whether ZF+CH or ZF+not-CH characterizes a broader notion of set. And a second way in which FBP-ists can salvage objective bite is by pointing out that certain mathematical disputes are disputes about whether some sentence is true in a *standard model*. (Balaguer 1995, p. 317)

Some of these correctives may also be dictated by naturalistic concerns: there are certain axioms which solve problems very nicely, or give us better pictures of the realm of sets, or are more elegant, concise and with a stronger 'unificatory' power.

As we have seen, Shelah admits that there are models which are more 'typical' than others, insofar as they would satisfy specific set-theoretic statements which, in turn, are more typical than others. Such statements he proposes to call 'semi-axioms'. The label is very aptly chosen, as it is supposed to convey the idea that none of these set-theoretic statements might eventually be viewed as an axiom. Shelah says:

> Generally, I do not think that the fact that a statement solves everything really nicely, even deeply, even being the best semi-axiom (if there is such a thing, which I doubt), is a sufficient reason to say that it is a "true" axiom. In particular, I do not find it compelling at all to see it as true. (Shelah 2003, p. 212)

[11] We say 'mild', as he seems to want to deny to be a fully committed (an 'extreme', in his words) formalist. He says: "..I reject also the extreme formalistic attitude which says that we just scribble symbols on paper or all consistent set theories are equal" (Shelah 2003, p. 212).

So we fittingly go back to 'unfettered' pluralism: although we may want to introduce a hierarchy of more or less convenient, of more or less typical set-theoretic statements, there is no hope to see any of these as more true of the set concept.

For the sake of completeness, we should mention that another way of being a non-realist multiverse view supporter might be that propounded by Feferman, that of a 'default', so to speak, pluralist. In Feferman's conception, the set concept is *vague*, and so is that of the 'linear' continuum. Consequently, the multiverse is a 'practically' inevitable construct, which will never be reduced to the simplicity of a single, definitive universe. The case for the solvability of such problems as CH is hopeless and, thus, any solution, in a sense, goes, insofar any solution is only a partial characterisation of both the set concept and of the continuum concept.

However, it should be noticed that Feferman is a 'default' pluralist only for truths concerning levels of the set-theoretic hierarchy beyond V_ω, and that, on the contrary, he attributes a full meaning to the whole of finite mathematics. Feferman's concerns, thus, are entirely different from the Carnapian's: the latter has no pre-existing theory of meaning, but only general criteria for adopting theories, whereas the Fefermanian pluralist is just a constructivist who is at a loss with the higher reaches of the set-theoretic hierarchy.[12]

2.3 The set-generic multiverse

2.3.1 Woodin

The next two multiverse views we are going to discuss are, in fact, one and the same conception, namely the *set-generic multiverse* view. However, as we shall see, they differ, to a certain extent, in their ultimate goal and in some other features. For instance, Woodin's conception leaves it open whether the set-generic multiverse view is at all plausible, and, in fact, its author suggests that this may not be the case. On the other hand, Steel argues that there is some significant evidence that truth in the set-generic multiverse could, ultimately, be reduced to truth in some simpler fragment of the multiverse itself, that is, its *core*. In the end, it would probably be more correct to describe the authors as universe view supporters, although their positions on this point are not always transparent.

In Woodin's case, our claim can be more strongly and convincingly substantiated. For instance, take the following crucial passage in Woodin (2011a), where the author sets forth his central epistemological view:

> It is a fairly common (informal) claim that the quest for truth about the universe of sets is analogous to the quest for truth about the physical universe. However, I am claiming an important distinction. While physicists would rejoice in the discovery that the conception of the physical universe reduces to the conception of some simple fragment or model, the Set Theorist rejects this possibility. I claim

[12] These views have been stated by the author several times. See, in particular, Feferman (1999), Feferman et al. (2000) and the more recent Feferman (2014). A careful response to Feferman's concerns is in Hauser (2002).

> that by the very nature of its conception, the set of all truths of the transfinite universe (the universe of sets) cannot be reduced to the set of truths of some explicit fragment of the universe of sets. [...] The latter is the basic position on which I shall base my arguments. (Woodin 2011a, pp. 103–104)

As we shall see, this philosophical position, as announced by the author, has some bearings on the 'decease' of the multiverse and, thus, orientates its construction from the beginning.

In very rough terms, the set-generic multiverse is generated by picking up a universe M from an initial multiverse ('the collection of possible universes of sets') and taking all generic extensions and refinements (inner models) of M. Suppose one starts with a countable transitive model M satisfying ZFC and let $\mathbb{V}_M$ the smallest set such that: 1) $M \in \mathbb{V}_M$; 2) for any M_1 and M_2, if M_1 is a model of ZFC and M_2 is a generic extension of M_1 and if either M_1 or M_2 are in $\mathbb{V}_M$, then both are in $\mathbb{V}_M$. We say that $\mathbb{V}_M$ is the set-generic multiverse generated in V from M.

As far as truth is concerned, it is natural to expect it to vary throughout the multiverse. As a matter of fact, there are some truths which hold in all set-generic extensions, that is, in all $N \in \mathbb{V}_M$. Suppose ϕ is one such truth: ϕ is, then, said to be a *multiverse truth*.

The *generic multiverse conception of truth* is the position that a sentence is true if and only if it holds in each universe of the generic multiverse generated by V. This can be formalized within V in the sense that for each sentence ϕ there is a sentence ϕ^* such that ϕ is true in each universe of the generic multiverse generated by V if and only if ϕ^* is true in V (Woodin 2011a, pp. 103–104).
The generic multiverse conception of truth is entirely reasonable from the point of view of a universe view supporter: truth *in* the multiverse ought to be defined as truth *in all members of* the multiverse, as long as truths holding in only one or some universes may not be seen as 'real' truths. It is not clear, however, that this position spells out a plausible multiverse view. After all, the multiverse was articulated precisely to make sense of our 'abundance' of truth, and, possibly, to understand what the reason for such an abundance was (e.g., by studying relationships among universes), whereas, by the generic multiverse conception of truth, Woodin's preoccupation, on the contrary, seems to be more that of bolstering a pre-multiverse attitude.

In fact, presumably, Woodin's idea from the beginning is that such constructs as set-generic extensions should not be taken as 'separately existing' constructs in the same way as Hamkins held (and we have seen that Hamkins' position may also be problematic). Furthermore, it is the very construction of the multiverse which makes appeal to a 'meta-universe', from which the multiverse is supposed to be 'generated' and this fact may be viewed as lending support to the generic multiverse conception of truth.

Now, Woodin can prove that, under certain conditions, into which we cannot delve here,[13] the set of the Π_2-multiverse truths is recursive in the set of the truths of V_{δ_0+1}, where δ_0 is the least Woodin cardinal.[14]

[13] Details on these can also be found in Woodin (2011a).

[14] For the definition of Woodin cardinals, see Kanamori (2003, p. 360).

This result, according to Woodin, violates the multiverse *laws*. These laws precisely prescribe that the set of Π_2-multiverse truths is not recursive in the set of the truths of V_{δ_0+1}. But then, just what is the rationale behind the multiverse laws? In plain terms, it is the aforementioned doctrine that no reduction of 'truth in V' to 'truth in a simpler fragment of V' is possible. Admittedly, the truths we are dealing with here are only a small fragment of the truths holding in V, so why should we ever bother formulating such a multiverse law? Because the multiverse analogue of 'truth in V', that is, of 'real' truths, as we said, is precisely 'truth in all members of the multiverse' and, among multiverse truths, Π_2-multiverse truths hold a special position, as they express that something is true at a certain V_α. In simpler terms, Π_2-multiverse truths would be the multiverse analogue of 'universe truths'.

Now, if the set of Π_2-multiverse truths is recursive in the set of truths of V_{δ_0+1}, then, there is a sense in which the set-generic multiverse fully captures 'truth in V'. But this, in turn, runs counter to Woodin's platonistic assumptions: set-theoretic truth cannot be reduced to truth in a certain fragment of V (that is, V_{δ_0+1}). This is, essentially, Woodin's argument for the rejection of the multiverse.

There may be legitimate reasons of concern about the philosophical tenability of the argument, but, even before that, it should be noticed that the argument rests upon some yet unverified mathematical hypotheses, and, therefore, until these hypotheses are not proved correct, in principle, it is not even known if such an argument can be produced. Some further, more general, concerns over the set-generic multiverse we will express in the next subsection, after examining Steel's framework.

2.3.2 Steel's programme

Unlike Woodin, Steel aims to articulate a *formal* theory of the set-generic multiverse. This is a first-order theory (**MV**), with two kinds of variables, one for *sets* and one for *worlds*. Within the theory, worlds are treated as proper classes and contain sets. In particular, one of the axioms of **MV** prescribes that an object is a set if and only if it belongs to some world.

Therefore, **MV** is a first-order theory which expands on ZFC, by specifying what models of ZFC constitute the multiverse of ZFC. In particular, worlds are either 'initial' worlds or *set-generic* extensions of initial worlds. The reason why we need a formal theory of the multiverse immediately reveals Steel's intents: a multiverse theory is construed first and foremost by Steel as a foundational theory, wherein one can develop both 'concrete' and set-theoretic mathematics. He says:

> ...we don't want everyone to have his own private mathematics. We want one framework theory, to be used by all, so that we can use each other's work. It's better for all our flowers to bloom in the same garden. If truly distinct frameworks emerged, the first order of business would be to unify them. (Steel 2012, p. 11)

But Steel also has another foundational concern, related to Gödel's programme. As known, the ZFC axioms are insufficient to solve lots of set-theoretic problems, and, what is worse, it is not clear what axioms one should adopt to solve them. Steel's concern is precisely that of finding an 'optimal' set theory extending ZFC, and he settles on *large cardinal axioms* as being the most 'natural' extensions of the ZFC axioms.

According to Steel, one optimal requisite of a natural extension T of ZFC is that of being able to *maximise* interpretative power, that is, of including and, possibly, extending the set of provable sentences of ZFC or of any of its extensions. There is a 'tool' we can use to see how theories perform in this respect: the linearly ordered scale of consistency strengths associated to axiomatic theories of the form ZFC+ large cardinals. It can be proved that, given any two examples of large cardinals, if H and T are two set theories containing them as axioms, then one invariably obtains that $H \leq_{Con} T$, $T \leq_{Con} H$ or $H \equiv_{Con} T$. In the first and second case, one says that T is stronger, consistency-wise, than H or viceversa, whereas, in the latter, H and T have the same consistency strength.

This gives us a linear arrangement of theories, for which a general interesting fact may hold: that is, if $H \leq_{Con} T$, then a fragment, and possibly all of, $Th(H)$ may be included in $Th(T)$. This holds, for instance, for statements at the level of arithmetic.[15] Now, given two theories H and T whose consistency strength is that of "infinitely many Woodin cardinals", then one may extend this result to second-order arithmetic. The hope is that, by strengthening the large cardinal assumptions associated to extensions of ZFC, the set of provable statements 'widens' accordingly. At the moment, this conjecture only holds in some specific mathematical structures.[16]

At any rate, by now it should be clear what Steel means by 'maximising interpretative power': a 'master' set theory should be one which, in a linear scale of theories, maximises over the set of provable statements. If the linear scale is given by the consistency strengths of 'natural' theories of the form ZFC+ large cardinals, such a maximisation is simply a function of the consistency strength of such theories. To recapitulate, using Steel's words:

> Maximizing interpretative power entails maximizing consistency strength, but it requires more, in that we want to be able to translate other theories/languages into our framework theory/language in such a way as to preserve their meaning. The way we interpret set theories today is to think of them as theories of inner models of generic extensions of models satisfying some large cardinal hypothesis, and this method has had amazing success. [...] It is natural then to build on this approach. (Steel 2012, p. 11)

But Steel has to confront a crucial problem: even within **MV** there is no hope to solve problems like CH. This is because, as known from the late 1960s, no large cardinal axiom solves CH. Therefore, Steel makes a step forward: we may want to reduce the complexity of **MV** by identifying a 'core' world within the framework we have set up. Steel preliminarily discusses two conceptions about the multiverse: *absolutism* and *relativism*, each coming with two different degrees of strength ('weak' and 'strong'). Relativism implies that there is no preferred world within **MV**, whereas absolutism

15 In fact, this result, as Steel clarifies, only holds for 'natural' theories, that is theories with 'natural' mathematical axioms, not quite like, for instance, the 'Rosser sentence'.

16 For instance, Steel's result on p. 7 only holds in $L(\mathbb{R})$.

identifies **MV** as only a transitory framework. Weak absolutism is what appeals most to Steel: there is a multiverse, but the multiverse has a core.

Weak absolutism aims to avail to itself two mathematical results. The first says that, if a multiverse has a definable world, then that world is unique and is included in all the others. The second is the existence of an axiom (Axiom **H**), which may be in line with the goal of maximising interpretative power in the way indicated and implies that the multiverse has a core.

If the axiom is true, then the multiverse of V has a core, which is, more or less, the HOD of any M which satisfies AD, the Axiom of Determinacy.[17] The axiom has many consequences and, in particular, implies CH.

2.3.3 Problems with the set-generic multiverse

We have already raised some concerns about Woodin's conception. We now want to review further potential objections, directed, this time, at both Woodin's and Steel's accounts.

First of all, the set-generic multiverse fosters a restrictive conception of the multiverse. As we have seen, Hamkins' radical viewpoint is consistent with an FBP-ist's presuppositions and ultimately depends on them for its justification. As far as the set-generic multiverse is concerned, the only grounds to accept it seem to be pre-eminently 'practical'. Woodin seems to be fully aware of this. In a revealing remark in his Woodin (2011a), he says:

> Arguably, the generic-multiverse view of truth is only viable for Π_2 sentences and not, in general, for Σ_2 sentences [...]. This is because of the restriction to *set forcing* in the definition of the generic multiverse. At present there is no reasonable candidate for the definition of an expanded version of the generic multiverse that allows for *class forcing* extensions and yet preserves the existence of large cardinals across the multiverse. (Woodin 2011a, p. 104)

Analogously, Steel motivates his ban on models other than set-generic extensions in **MV** in the following manner:

> Our multiverse is an equivalence class of worlds under "has the same information". Definable inner models and sets may lose information, and we do not wish to obscure the original information level. For the same reason, our multiverse does not include class-generic extensions of the worlds. There seems to be no way to do this without losing track of the information in what we are now regarding as the multiverse, no expanded multiverse whose theory might serve as a foundation. We seem to lose interpretative power. (Steel 2012, p. 13)

From both quotations, it seems clear that the authors' main concern is about losing large cardinals and their associated interpretative power, as class forcing does not, in general, *preserve* large cardinals. In particular, Steel's reasons to cling on large cardinals seemed to be well-motivated, in the light of his foundational programme.

[17] For more accurate mathematical details, we refer the reader to Steel's cited paper.

However, restrictiveness remains a patent weakness of the set-generic multiverse view, and one which cannot be easily repaired: in our view, the authors propose it essentially because, otherwise, they could not obtain the results they are most interested in, that is 'unification' via the core hypothesis (Steel) or the retreat to a universe view (Woodin).

It should be noticed that Steel himself has expressed some significant criticisms of Woodin's conceptual framework. In a footnote, he says:

> ...the decision to stay within the multiverse language does not commit one to a view as to what the multiverse looks like. The "multiverse laws" do not follow from the weak relativist thesis. The argument that they do is based on truncating worlds at their least Woodin cardinal. However, this leaves one with nothing, an unstructured collection of sets with no theory. (Steel 2012, p. 16)

He continues:

> The Ω-conjecture does not imply a paradoxical reduction of **MV** or its language to something simpler, because there is no simpler language or theory describing a "reduced multiverse". (*ibid.*)

As a matter of fact, Steel's criticism may well apply to his own framework: the preference for large cardinals as natural axioms also yields a 'reduced' multiverse, one where, presumably, there is no universe which does not contain large cardinals.

One further reason of concern with Steel's account of the multiverse may be raised by the staunch multiverse view supporter with respect to the purpose of 'unification'. Steel owes us a more coherent explanation of why **MV** with a core would be more suitable to our purposes than **MV** with no core. Steel's response would, presumably, be that in the former case we increase our level of information, by enlarging the set of provable statements. But at what price? For instance, one may still hold the legitimate view that CH is false, in the face of its being true in the core. Steel's way out of this is to ultimately appeal to naturalistic concerns, which would override all other sorts of concerns about truth. He says:

> The strong absolutist who believes that V does not satisfy CH must still face the question whether the multiverse has a core satisfying Axiom **H**. If he agrees that it does, then the argument between him and someone who accepts Axiom **H** as a strong absolutist seems to have little practical importance. (Steel 2012, p. 17)

What Steel seems to suggest here is that even the strong absolutist who believes that CH is false must acknowledge that the multiverse has a core and, as a consequence of this, he might come to hold the truth of CH. Now, it may well be that the strong absolutist might come to accept the truth of CH via the acceptance of the Axiom **H**, that is through accepting **MV** with a core for 'extrinsic' reasons, but then, in turn, it would be the Axiom **H** which would be in strong need of that kind of justification the strong absolutist would be more naturally inclined to accept. In other terms, it is far from obvious that all strong absolutists would come to accept the Axiom **H** on purely naturalistic grounds. They might want to have some stronger justification, probably stronger than the one Steel can, at present, offer them.

3 The 'vertical' multiverse

In this section, we first discuss the actualist and potentialist conceptions of V and then examine Zermelo's account of the universe of sets, which contains features of both conceptions. Our aim is to argue that Zermelo's account can be viewed as one further multiverse conception (featuring a 'vertical' multiverse), which allows V to be heightened while keeping its width fixed. As said in the beginning (Sect. 1.1), this multiverse conception is preferable for the implementation of the hyperuniverse programme, whose general features and goals we briefly review in Sect. 3.3.1.

Finally, in the last subsection, we shall show how an infinitary logic (V-logic) can be used to express the horizontal maximality of the 'vertical' multiverse.

3.1 Actualism and potentialism

We take actualism as a position which construes V as an actual object, a fully actualised domain of all sets, as something given which, accordingly, cannot be modified. According to actualists, there is no way to 'stretch' V: any model-theoretic construct which seems to do this produces, in fact, a construction which is within V. We saw that Koellner's criticism of the Hamkinsian multiversist, ultimately, seemed to advocate such a view: all the universes Hamkins thought to exist separately did not require of one to conceive anything more than V and, thus, an actualist can still accommodate them to her conception.

A potentialist, on the other hand, sees V as an indefinite object, which can never be thought of as a 'fixed' entity. The potentialist may well believe that there are some fixed features of V, but she believes that these are not sufficient to fully make sense of an 'unmodifiable' V: the potentialist believes that V is indeed 'modifiable' in some sense.

The actualist/potentialist dichotomy seems to recapitulate a great part of the current philosophy of set theory, insofar as one of the latter's primary concerns is to make sense of the nature of V. Actualists are more naturally grouped with realists, whereas potentialists with non-realists. However, this does not have to be necessarily the case. For instance, one may be a potentialist about classes (like V itself) and nonetheless be a realist. That V is not a set we know from the early emergence of the paradoxes, but just what else it should be is unclear and, in fact, early set theory dealt with this issue from diverse angles.[18]

It seems natural to distinguish the following four types of actualism and potentialism[19]:

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[18] A crucial reading on this is Hallett's book on the emergence of the *limitation of size* doctrine, Hallett (1984). The distinction between actualists and potentialists may be construed as the result of different interpretations of Cantor's *absolute infinite*. One of the most exhaustive articles on Cantor's conception of absoluteness and of its inherent tension between actualism and potentialism is Jané (1995). For a discussion of actualism and potentialism, with reference to the justification of *reflection principles*, see Koellner (2009a). For an accurate overview of several potentialist positions, see Linnebo (2013).

[19] The distinction between 'height' and 'width' of the universe is firmly based on the iterative concept of set: the length of the ordinal sequence determines the height of the universe, while the width of the universe is given by the powerset operation.

HEIGHT ACTUALISM: the height of V is fixed, that is, no new ordinals can be added,

WIDTH ACTUALISM: the width of V is fixed, that is no new subsets can be added.

HEIGHT POTENTIALISM: the height of V is not fixed, new ordinals can always be added,

WIDTH POTENTIALISM: the width of V is not fixed, new subsets can always be added.

One could also hold a 'mixed' position, that is, one could be a height potentialist and a width actualist or a width potentialist and a height actualist. While a priori a mixed position might seem less tenable, we will argue in the next section that the Zermelian concept of set naturally leads to such a position.

Another crucial factor which bears on the distinction between actualism and potentialism and the preference for one over the other is a concern for the 'maximality of V'. The 'maximality of V' seems to adumbrate the possibility that V be conceived as one among many objects, as a 'picture' among other 'pictures' of the universe. If one is a potentialist, then one can more easily make sense of different 'pictures' of V. In particular, one can make sense of the stretching of V, that is, its being 'extended' in height and width, in ways which are seen to be maximal in some respect. We sometimes call such extensions 'lengthenings' (when new ordinals are added) and 'thickenings' (when new subsets are added).

3.2 Zermelo's account: the 'vertical' multiverse

Our examination of Zermelo's views is based on his Zermelo (1930). In this paper, Zermelo shows that the axioms of second-order set theory Z_2 are *quasi-categorical* it the sense that every model M of Z_2 is of the form $(V_\kappa, \in)$ where κ is a strongly inaccessible cardinal; V_κ is called a *natural domain*.

Zermelo construes the sequence of natural domains 'dynamically', as the unfolding of a temporary, endless actualisation of the universe. However, if one looks closer, within this process, there is no longer a single universe present. The universe V, in this construction, becomes just a collection of different V_α's whose width is fixed, and whose height can be extended.

Zermelo vividly recapitulates his approach in the following manner:

> To the unbounded series of Cantor ordinals there corresponds a similarly unbounded double-series of essentially different set-theoretic models, in each of which the whole classical theory is expressed. The two polar opposite tendencies of the thinking spirit, the idea of creative advance and that of collection and completion [*Abschluss*], ideas which also lie behind the Kantian 'antinomies', find their symbolic representation and their symbolic reconciliation in the transfinite number series based on the concept of well-ordering. This series reaches no true completion in its unrestricted advance, but possesses only relative stopping-points, just those 'boundary numbers' [*Grenzzahlen*] which separate the higher model types from the lower. Thus the set-theoretic 'antinomies', when correctly understood, do not lead to a cramping and mutilation of mathematical science, but rather to an, as yet, unsurveyable unfolding and enriching of that science. (Zermelo (1930), in Ewald (1996, p. 1233))

Zermelo's natural models can be viewed as a tower-like multiverse, where each universe is indexed by an inaccessible cardinal. If we have a proper class of inaccessible cardinals, then every natural model can be extended to a higher natural model: in our current terminology, the Zermelian concept of set theory is an example of *potentialism in height*.

On the other hand, as said, Zermelo's account is second-order, insofar as it seems to adumbrate the availability of a collection of 'definite' properties of sets (over which the axioms and, in particular, the Axiom of Separation, quantify) and this, in turn, implies the fixedness of the power-set operation. So, Zermelo's account also constitutes an example of *actualism in width*.

To sum up, Zermelo's conception of V can be seen as a multiverse conception that features a 'vertical' multiverse which embraces height potentialism and width actualism.

Now, Zermelo's conception seems preferable for the hyperuniverse programme because it entirely befits its goals, that is the search for optimal mathematical principles expressing the maximality of V, and we now explain why.

There are two main forms of maximality: maximality in height (*vertical maximality*) and maximality in width (*horizontal maximality*). Vertical maximality can be formulated in many ways, but perhaps its ultimate form was introduced and studied in Feferman and Honzik (forthcoming) by two of the present authors. Such a principle is in line with the height potentialism inherent in Zermelo's account.[20]

As far as horizontal maximality is concerned, it would seem that width potentialism would best suit the goals of the hyperuniverse programme. However, we feel there is no need to drop Zermelo's width actualism, but the reasons for our choice are different from Zermelo's. As said above, Zermelo committed to a second-order version of the axioms, which implied the determinacy of the power-set operation. We do not hold this position. In our view, height potentialism is natural in light of the clear and coherent way one can add new V-levels by extending the ordinal numbers through iteration, whereas width potentialism is not, as there is no analogous clear and coherent iteration process for enlarging power-sets. In other terms, the addition of ordinals can be carried out in an orderly, stage-like manner, whereas the addition of subsets cannot. This is the reason why we accept Zermelo's width actualism in our account of V, and, as we shall show in Sect. 3.3.2, there is a way of exploring horizontal maximality which makes Zermelo's account fully compatible with the hyperuniverse programme. On the other hand, as explained above, height potentialism is entirely natural, and, moreover, within the hyperuniverse programme, height actualism puts severe restrictions on formulations of the maximality of V.

We mentioned the programme several times. It is now time to provide the reader with more details about the programme. In Sect. 3.3 we briefly review it and then describe recent results which make sense of a theory of horizontal maximality.

[20] Also Reinhardt's theory of *legitimate candidates* (see Reinhardt (1974)) seems to follow Zermelo's account. Finally, Hellman (in Hellman (1989)), develops a structuralist account of the universe in line with Zermelo's concerns, but based on modal assumptions.

3.3 The 'vertical' multiverse within the hyperuniverse programme

3.3.1 A brief review of the hyperuniverse programme

The programme was introduced by the second author and Arrigoni in Arrigoni and Friedman (2013). In that paper, the word *hyperuniverse* was introduced to denote the collection of all transitive countable models of ZFC. Within the programme, such models are viewed as a technical tool allowing set-theorists to use the standard model-theoretic and forcing techniques (the Omitting Types Theorem and the existence of generic extensions, respectively). The underlying idea is that the study of the members of the hyperuniverse allows one to indirectly examine properties of the real universe V (that we construe, by now, as the 'vertical' multiverse discussed above).

The hyperuniverse programme is essentially concerned with the notion of the *maximality* of the universe. As already mentioned above, we construe the 'maximality of V' as implying that V should be maximal among its different 'pictures', that is, candidates for universes. It is precisely here that the hyperuniverse is helpful, because it provides the context, i.e., the collection of candidates, where we can look for a maximal universe.

Thus the hyperuniverse programme analyses the maximality of V through the study of non-first-order maximality properties of members of the hyperuniverse. Among the earliest examples of such properties is the IMH, the Inner Model Hypothesis:

Definition (*IMH*) Let M be a member of the hyperuniverse. For any φ, whenever there is an outer model W of M where φ holds, there is a definable inner model $M' \subseteq M$ where φ also holds.

This principle clearly postulates that M is 'maximal' in *width*, in some sense, among the members of the hyperuniverse. Using large cardinals, one can prove the consistency of IMH.[21]

The programme is based on the conviction—still to be verified—that the different formalisations of the notion of maximality will lead to an optimal such formalisation, and that the first-order sentences which hold in all universes which exhibit that optimal form of maximality will be regarded as first-order consequences of the notion of the maximality of V and, therefore, as good candidates for new axioms of set theory. The programme also aims to make a case for the 'intrinsicness' of such axiom candidates, insofar as their selection, in the end, would only depend upon a thorough analysis of the concept of the 'maximality of V'.

Now, just what is the connection between the hyperuniverse and the 'vertical' multiverse described above? We view the hyperuniverse only as a technical tool. However, one may also view it as one further 'auxiliary' multiverse, more suited to the kind of mathematical investigations to be carried out within the programme. In this sense, one could say that the 'vertical' multiverse is supplemented—for strictly mathematical reasons—by such an auxiliary multiverse.

21 The proof is in Friedman et al. (2008).

3.3.2 Width actualism and infinitary logic

We now proceed to present mathematical results which show that one can address horizontal maximality within the 'vertical' multiverse, that is, in a V which is potential in height but actual in width, by using the hyperuniverse as a mathematical tool.

Let us, first, briefly summarise in conceptual terms what we will, then, be showing in a rigorous mathematical fashion. In the hyperuniverse programme we refer to both 'lengthenings' and 'thickenings' of V and, in particular, the IMH contains a reference to 'thickenings'. 'Lengthenings' are entirely unproblematic for a height potentialist, whereas 'thickenings' would appear to collide with the width actualism inherent in Zermelo's account we chose to adopt. However, we argue that, using V-logic, we can express properties of 'thickenings' of V without actually requiring the existence of such 'thickenings', and, moreover, these properties are first-order over what we call Hyp(V), a modest 'lengthening' of V. Therefore, with the mathematical tools we will be employing, there is no violation of the Zermelian conception to which we commit ourselves.

We start by noticing that the Löwenheim–Skolem theorem allows one to argue that any first-order property of V reflects to a countable transitive model (that is, a member of the hyperuniverse). However, on a closer look, one needs to deal with the problem that not all relevant properties of V are first-order over V; in particular, the property of V 'having an outer model (a 'thickening') with some first-order property' is a higher-order property. We show now that, with a little care, all reasonable properties of V formulated with reference to outer models are actually first-order over a slight extension ('lengthening') of V.[22]

We first have to recall some basic notions regarding the infinitary logic $L_{\kappa,\omega}$, where κ is a regular cardinal.[23] For our purposes, the language is composed of κ-many variables, up to κ-many constants, symb ols $\{=, \in\}$, and auxiliary symbols. Formulas in $L_{\kappa,\omega}$ are defined by induction: (i) All first-order formulas are in $L_{\kappa,\omega}$; (ii) whenever $\{\varphi\}_{i<\mu}$, $\mu < \kappa$ is a system of formulas in $L_{\kappa,\omega}$ such that there are only finitely many free variables in these formulas taken together, then the infinite conjunction $\bigwedge_{i<\mu} \varphi_i$ and the infinite disjunction $\bigvee_{i<\mu} \varphi_i$ are formulas in $L_{\kappa,\omega}$; (iii) if φ is in $L_{\kappa,\omega}$, then its negation and its universal closure are in $L_{\kappa,\omega}$. Barwise developed the notion of proof for $L_{\kappa,\omega}$ and showed that this syntax is complete, when $\kappa = \omega_1$, with respect to the semantics (see discussion below and Theorem 3.3.2).

[22] The use of the Löwenheim–Skolem theorem, while completely legitimate, is actually optional: if one wishes to analyse the outer models of V without 'going countable', one can do it by using the V-logic introduced below. However, there is a price to pay: instead of having the elegant clarity of countable models, one will just have to refer to different theories. This has analogies in forcing: to have an actual generic extension, one needs to start with a countable model; if the initial model is larger, one can still deal with forcing syntactically, but a generic extension may not exist (see, for instance, Kunen (2011)). A more relevant analogy in our case is that the Omitting Types Theorem (which is behind V-logic) works for countable theories, but not necessarily for larger cardinalities.

[23] Full mathematical details are in Barwise (1975). We wish to stress that the infinitary logic discussed in this section appears only at the level of theory as a tool for discussing outer models. The ambient axioms of ZFC are still formulated in the usual first-order language.

A special case of $L_{\kappa,\omega}$ is the so-called V*-logic*. Suppose V is a transitive set of size κ. Consider the logic $L_{\kappa^+,\omega}$, augmented by κ-many constants $\{\bar{a}_i\}_{i<\kappa}$ for all the elements a_i in V. In this logic, one can write a single infinitary sentence which ensures that if M is a model of this sentence (which is set up to ensure some desirable property of M), then M is an outer model of V (satisfying that desirable property). Now, the crucial point is the following: if V is countable, and this sentence is consistent in the sense of Barwise, then such an M really exists in the ambient universe.[24] However, if V is uncountable, the model itself may not exist in the ambient universe, but, in that case, we still have the option of staying with the syntactical notion of a consistent sentence.

We have to introduce one further ingredient, that of an *admissible set*. M is an admissible set if it models some very weak fragment of ZFC, called Kripke-Platek set theory, KP. What is important for us here is that for any set N, there is a smallest admissible set M which contains N as an element—M is of the form $L_\alpha(N)$ for the least α such that M satisfies KP. We denote this M as $\mathrm{Hyp}(N)$.

And now for the following crucial result:

Theorem (Barwise) *Let V be a transitive set model of ZFC. Let $T \in V$ be a first-order theory extending ZFC. Then there is an infinitary sentence $\varphi_{T,V}$ in V-logic such that following are equivalent:*

1. *$\varphi_{T,V}$ is consistent.*
2. $\mathrm{Hyp}(V) \models$ "*$\varphi_{T,V}$ is consistent*."
3. *If V is countable, then there is an outer model M of V which satisfies T.*

By the theorem above, if we wish to talk about outer models of V ('thickenings'), we can do it in $\mathrm{Hyp}(V)$—a slight lengthening of V—by means of theories, without the need to really thicken our V (and indeed, we cannot thicken it if we are width actualists). However, if we wish to have models of the resulting consistent theories, then, using the Löwenheim–Skolem theorem, we can shift to countable transitive models. And this is precisely where the hyperuniverse comes into play.[25]

Now, we also want to make sure that members of the hyperuniverse really witness statements expressing the horizontal maximality of V. One such statement was the mentioned IMH.

Recall that V satisfies IMH if for every first-order sentence ψ, if ψ is satisfied in some outer model W of V, then there is a definable inner model $V' \subseteq V$ satisfying ψ. Ostensibly, the formulation of IMH requires the reference to all outer models of V, but with the use of infinitary logic, we can formulate IMH syntactically in $\mathrm{Hyp}(V)$ as follows: V satisfies IMH if for every $T = \mathrm{ZFC} + \psi$, if $\varphi_{T,V}$ from Theorem 3.3.2 above is consistent in $\mathrm{Hyp}(V)$, then there is an inner model of V which satisfies T. Finally, with an application of the Löwenheim–Skolem theorem to $\mathrm{Hyp}(V)$, this becomes a statement about elements of the hyperuniverse.

24 Again, for more details we refer the reader to Barwise (1975).

25 This is in clear analogy to the treatment of set-forcing, see Footnote 22. However, note that unlike in set-forcing, where the syntactical treatment can be formulated inside V, to capture arbitrary outer models, we need a bit more, i.e. $\mathrm{Hyp}(V)$.

4 Concluding summary

We have seen that the multiverse construct can be spelt out in different ways. We proposed a unified way to classify different positions, centered on the realism/non-realism conceptual dichotomy.

The realist multiverse view was represented by the Balaguer–Hamkins multiverse, wherein different universes instantiated different concepts of set (or, alternatively, different models instantiated different collections of axioms), whereas the non-realist multiverse view (whose main representative was Shelah) was construed as a form of radical pluralism with no explicit commitment to an ontological position. The other two conceptions we reviewed (Woodin's and Steel's) contribute to the discussion about the multiverse in the following manner: they explore a limited, but mathematically rich, concept of the set-theoretic multiverse, that consisting of a collection of set-generic extensions.

Finally, in the last section, building on the Zermelian account of V, with its conceptual reliance on height potentialism and width actualism, we have described one further multiverse conception that features a 'vertical' multiverse.

We believe that this conception—along with the use of infinitary logic and the hyperuniverse as 'auxiliary' multiverse—is the preferable multiverse conception for the hyperuniverse programme, insofar as: (1) it allows one to formulate maximality principles addressing 'lengthenings' and 'thickenings' of the universe (for 'thickenings', though, we also need to use V-logic), and, at the same time, (2) it does not compel us to embrace the potentiality of the power-set operation, for which, as discussed earlier, we do not have a clear and conceptually satisfying framework. In our view, the fact that the hyperuniverse programme exhibits the potential for generating new axioms of set theory through the study of maximality can be used as an argument in favour of this multiverse conception.

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