Intrinsic Justification for Large Cardinals and Structural Reflection

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Main Questions and Issues

Question 1

Are large cardinals (LC) *intrinsically justified*? (rough translation: are they *true* principles of set theory?)

Question 2

Does intrinsic justification for LC (if there is such a thing) *extend* to *all* of them?

This has led us to identify two fundamental issues:

Intrinsicness Issue: LC axioms are *true* of the concept of set.

Universality Issue: LC axioms can *all* be captured by a single (possibly, intrinsically justified) (set of) mathematical principle(s).

Main Hurdles

Concept of Set = Iterative Concept of Set (ICS)

Set theory is *just* about entities, *sets*, which are *iteratively* and *cumulatively* formed in *stages* (indexed by the *ordinals*).

Schindler's Challenge (cf. [Schindler, 1994], and [McCallum, 2021])

Take ICS to be the only correct explication of the concept of set. Then, LC are not true of ICS, since they are equivalent to principles (Bernays' Reflection) which imply the existence of other entities (*impredicative* proper classes).

Main Hurdles/Continued.

Fact (cf. [Koellner, 2009])

The main (potentially, *intrinsically justified*) methodology (principle) to produce large cardinals, the Reflection Principle (or strengthenings thereof), is not able to produce large cardinals *incompatible with* $V = L$.

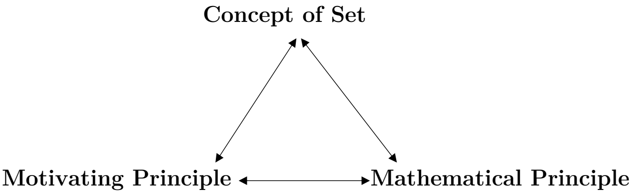
Koellner's Challenge (cf. [Koellner, 2009])

If the Reflection Principle (or strengthenings thereof) is the only principle motivating large cardinals compatible with ICS, then it is not possible to find *intrinsic* motivation for LC which break the $V = L$ barrier.

The Strategy

- Intrinsic justification for LC is (indirectly) provided by 'abstract motivating principles' (such as Reflection).
- Review all fundamental *abstract motivating principles*.
- Examine mathematical expressions of such principles.
- Choose, among these, expressions which are closest to being seen as workable responses to Schindler's and Koellner's Challenges (solutions to the *Intrinsicness* and *Universality Issue*).
- If possible, buttress justification of mathematical principles through isolating further (not necessarily *intrinsic*) *evidence* for them

The Picture


Concept of Set
Motivating Principle
Mathematical Principle

Gödel's Big Five plus Resemblance (Richness)

[Wang, 1996], [Kanamori and Magidor, 1978], [Solovay et al., 1978], [Maddy, 1988a] etc.:

- *Extensionalisation*
- *Closure*
- *Intuitive Range*
- *Reflection* – The universe of sets is *structurally indefinable*.
- *Uniformity (of the universe of sets)* – Properties of sets *reappear* over and over in V.
- *Resemblance* – There are mutually *indistinguishable* rank initial segments of V.[1]

[1] *Richness*: The universe of sets is so *rich* that there always exist *distinct* ordinals α and β such that V_α and V_β are *indistinguishable* ([Maddy, 1988b], but cf. also [Martin, 1976]).

On Abstract Motivating Principles

..Gödel formulated a summary of the five principles actually used for setting up axioms of set theory: (1) intuitive range; (2) closure principle; (3) reflection principle; (4) extensionalization; and (5) uniformity. He emphasized that the same axiom can be justified by different principles, which are nonetheless distinct because they are based on different ideas—for example, inaccessible numbers are justified by either (2) or (3). ([Wang, 1996], p. 280)

In tracing the development of the theory of large cardinals, we will emphasize the interplay of three major themes. The first of these is the role of those abstract motivating principles which have led to the formulation of large cardinal properties.
([Kanamori and Magidor, 1978], p. 104)

Reflection

Reflection Principle

For any property P of V,[a] there is an ordinal α s.t. $V_\alpha \models P^\alpha$.

[a] Expressed by a *formula* in the set-theoretic language.

Reflection Theorem Schema (ZF)

$(\exists\alpha)(\forall x_1, \ldots, x_n \in V_\alpha)(\phi(x_1, \ldots, x_n) \leftrightarrow \phi^{V_\alpha}(x_1, \ldots, x_n))$.

Second-Order Reflection (Bernays' Reflection)

$\Phi \to (\exists u)(Trans(u) \wedge \Phi^u)$, for all Π^1_1-formulae Φ.

Through Second-Order Reflection, one can define *inaccessible*, *Mahlo*, *indescribable*, *weakly compact* cardinals, etc.

Problems with Reflection

- Further strengthenings may be *inconsistent* or *weak* (justification of LC only up to the level of *Erdős cardinals*), hence Koellner's Challenge. ([Koellner, 2009])
- As shown by [Schindler, 1994], Barnays' Reflection *already* implies the existence of *impredicative* classes (hence Schindler's Challenge).
- [Tait, 1998] aims to show that large cardinals consistent with $V = L$ are consistent with ICS through his 'bottom-up' approach. Problems:
    1. Reflection restricted to only certain classes of formulas (*positive* formulas).
    2. Approach is too general (and prone to inconsistency).
    3. Also [McCallum, 2021]'s extension of Tait's approach may have the same problems (?).

Resemblance through Elementary Embeddings

Resemblance is taken to be expressible through *elementary embeddings:*

..The next conceptual step is to say that there are elementary embeddings $(V_\alpha, \in) \to (V_\beta, \in)$. ([Solovay et al., 1978], p.75)

As to the connection between Reflection and Resemblance:

To see the reflection inherent in such a principle, notice that if P is a property of $\kappa = crit(j)$*, and M resembles V enough that* $P(\kappa)$ *is true in M, then M satisfies* $(\exists \alpha < j(\kappa))P(\alpha)$*, so that if j is elementary with respect to P, V satisfies* $(\exists \alpha < \kappa)P(\alpha)$*.* ([Martin and Steel, 1989], p. 73

Principles

- (ZA). [Reinhardt, 1974]'s S4 axiom:

$$\exists j : V_{\alpha_0+1} \to V_{\alpha_1+1}$$

  j an e.e., V_{α_0} and V_{α_1} r.i. segments of V, and $\alpha_1 > \alpha_0$. Then, $crit(j) = \alpha_0$ is a *1-extendible cardinal*.
- (NBG). [Welch, 2012]'s Global Reflection Principle:

$$\exists j : (V_\kappa, \in, D) \to (V, \in, \mathcal{C})$$

  j an e.e., $D \subseteq V_{\kappa+1}$, $\mathcal{C}$ is the collection of all classes of V. Then, $crit(j) = \kappa$ is a *measurable* and *Woodin cardinal*.
- So, both are are very strong!: they clearly break the $V = L$ barrier.

Uniformity *meets* Resemblance

Ackermann's Reflection (according to Gödel)

The Absolute is unknowable.

Reinhardt's Reflection

The *properties* of the Absolute are unknowable.

- Reinhardt's S4 can be strengthened by extending Reflection to 3rd-order predicates, 4th-order predicates, etc. (all of [Reinhardt, 1974]'s Ω-classes).
- Cf. [Marshall, 1989]'s theory with Reflection for 1-classes and 2-classes.

So, Resemblance principles may, in the end, also embody *Uniformity*.

Troubles with Resemblance

- There's a problem with classes as *intensional* notions (problem of 'tracking' classes, as explained in [Koellner, 2009]).
- Problem with *arbitrary* classes (incompatible with ICS): in S4 and $\mathrm{GRP}_{\Sigma^1_\infty}$.
- Problem with proper classes taken as *determinate existents* (cf. [Horsten and Welch, 2016]).
- LC axioms are 'picked out' unsystematically: no general methodology to address the *Universality Issue* is provided.

Structural Reflection

Structural Reflection (SR), [Bagaria, 2023], p. 30

For every *definable*, in the first-order language of set theory, possibly with parameters, class $\mathcal{C}$ of relational structures of the same type there exists an ordinal α that *reflects* $\mathcal{C}$, i.e., for every A in $\mathcal{C}$ there exists B in $\mathcal{C} \cap V_\alpha$ and an *elementary embedding* from B into A.

In a sense, SR could be seen as a strengthening and expansion of:

Vopěnka's principle

For every proper class of structures of the same type there are two $A \neq B$ such that A is *embeddable into* B.

Pinning Down Structural Properties

SR is motivated by the notion of the:

Indefinability of V (Gödel in [Wang, 1996])

V is *not definable* by any *internal structural property* of the membership relation.

This was connected by Gödel to the Reflection Principle itself.

SR identifies structural properties with such structures as: $\langle A, \in, \langle R_i \rangle_{i \in I} \rangle$, with I possibly empty, and A a set.

There is, then, a thick sense in which SR is also an elucidation of Reflection.

Structural Reflection (Cont'd).

SR will split into different SRPrinciples, according to the complexity of the *definability class* Γ (and relational formula Φ) associated to it.

Σ_n-SR

For each natural number n, for every Σ_n-definable class $\mathcal{C}$ of structures of the same type there is an ordinal α that *reflects* $\mathcal{C}$.

Π_n-SR

For each natural number n, for every Π_n-definable class $\mathcal{C}$ of structures of the same type there is an ordinal α that *reflects* $\mathcal{C}$.

(Note: these also have boldface versions (with parameters), $\boldsymbol{\Sigma_n}$-SR and $\boldsymbol{\Pi_n}$-SR).

SRPs and LC

Just to give an idea of how strong SRPs are, consider:

- Π_1-SR (Σ_2-SR) is equivalent to the existence of a *supercompact* cardinal.
- Π_2-SR (Σ_3-SR) is equivalent to the existence of an *extendible* cardinal.
- (for all n) Π_n-SR is equivalent to VP (taken as a *schema*).

SRPs and LC/Cont'd.

Refinements of basic SR have been shown to be able to deal with fragments of the spectrum of consistency strengths of LC (cf. [Bagaria, 2023] and [Bagaria and Lücke, 2023]). E.g.:

- Γ-SR – from *supercompact* to *beyond extendible* (up to the level of Vopěnka's Principle);
- SR^- – *small* LC (*inaccessible, Mahlo, weakly compact*, etc.)
- ProductSR – *strong* and *Woodin*;
- StrongGenericSR – *remarkable cardinals*
- ExactSR – LC beyond Vopěnka's Principle
- ...

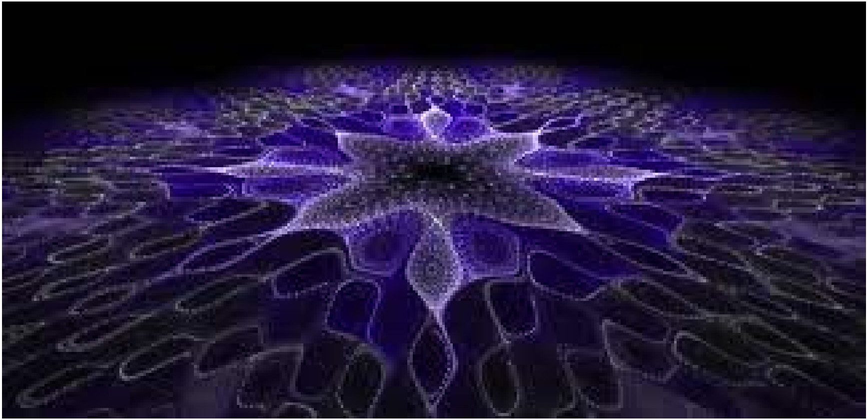
Conjecture. *Every known large cardinal notion is equivalent to some SRP.*

Justifying Large Cardinals Through SRP's

- SRPs may be able to incorporate *all* large cardinals (hence, yield a response to the *Universality Issue*).
- The amount of class theory required for SRPs is 'modest' (all classes in SRPs are ZFC-*definable* classes, hence, SRPs may be compatible with ICS (Schindler's Challenge met).
- (Wrt this, also note that the embeddings used in SRPs are just *sets*!)
- SR also seems to be intrinsically backed by a mix of *Reflection*, *Resemblance* and *Richness* (but NB. ICS may still not be sufficient to justify the use of *elementary embeddings* in SR!)
- Claim: assuming the truth of SRPs, a way to (partly) meet Koellner's Challenge is available, and (tentatively) *Intrinsicness Issue* could be overcome.

End Slide

Thanks!



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