Higher-Order Platonism and Multiversism

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Abstract

Joel Hamkins has described his multiverse position as being one of 'higher-order realism – Platonism about universes', whereby one takes *models of set theory* to be actually existing objects (vis-à-vis 'first-order realism', which takes only *sets* to be actually existing objects). My goal in this paper is to make sense of the view in the very context of Hamkins' own multiversism. To this end, I will review what may be considered the central features of higher-order platonism and some of its potential interpretations, and then will focus on Zalta and Linsky's Object Theory, which, I will argue, bears the best approximation to Hamkins' conception. I will then show how the embedding of higher-order platonism into Object Theory may help the Hamkinsian multiversist to respond to salient criticisms of the multiverse conception, especially those relating to its articulation, skeptical attitude, and relationship with set-theoretic practice.

1 Hamkins' Multiverse

The 'set-theoretic multiverse conception' holds that the subject matter of set theory is not a *single* universe (in particular, not the well-founded, cumulative hierarchy V), but, in fact, *some* (or *all*) the *models* of some set theory T (most commonly, of ZFC).[1]

The most lucid and radical expression of this conception can be found in [Hamkins, 2012]. This is not to deny that alternative conceptions are possible (and, indeed, have been formulated); only, Hamkins' multiversism seems to stand out as the most coherent, and self-conscious, form of *set-theoretic pluralism*.[2]

Despite having the unquestionable merit of bringing about the emergence of a new approach to set-theoretic foundations, [Hamkins, 2012] plainly rests on philosophically controversial claims. The invocation of the existence of a *plethora* of universes of set theory looks especially controversial.

[1] I assume that readers have some familiarity with 'models of set theory'. For a systematic treatment, which covers all models mentioned in this paper, see [Kunen, 2011].

[2] The literature on 'multiverse conceptions' is vast. A rough classification, and discussion of the main features of each conception, may be found in [Antos et al., 2015]. An introduction to, and discussion of, the basics of 'set-theoretic pluralism' may be found in [Linnebo, 2017].

Moreover, as a preliminary characterisation of the philosophy it aims to advocate throughout, the paper makes it clear, from the beginning, that the 'multiverse position' is:

> ..one of higher-order realism—Platonism about universes—and I defend it as a realist position asserting actual existence of the alternative set-theoretic universes into which our mathematical tools have allowed us to glimpse. The multiverse view, therefore, does not reduce via proof to a brand of formalism. In particular, we may prefer some of the universes in the multiverse to others, and there is no obligation to consider them all as somehow equal.
> ([Hamkins, 2012], p. 417)

In a fully consequential manner, a substantial amount of work is, then, devoted to articulating the idea that *set-theoretic models*, in particular, models obtained through *forcing*, may be construed as 'determinate objects' in the *platonic* sense.[3]

The goal of this paper is to investigate whether and how Hamkins' multiverse conception may legitimately claim to be able to do that, and on what metaphysical grounds; a parallel question, which I will also investigate, is that of how well 'higher-order realism' serves Hamkins' own purpose of laying out a *pluralist* foundation of set theory.

In order to be clear about exactly what is the position I shall be dealing with, I shall first explicate what can be viewed as the two fundamental tenets of Hamkins' conception, which I shall henceforth indicate as Higher-Order Platonism (HOP). We may summarise them as follows:

Platonism (PLAT). The models of set theory are *platonic* entities, i.e., are self-standing, independently existing, physically acausal, mathematical objects.

Perspectivism (PERSP). The set-theoretic multiverse is always *relative to* a single model, that is, it is *describable* only from the point of view of one particular model.

Commentators lay a lot of emphasis on (PLAT), but I am inclined to see (PERSP) as being even more crucial to HOP's purposes. Some further explication of (PERSP) may, therefore, already be useful at this stage.

The view implies that set-theoretic structures are not definable ('observable') from the point of view of an absolute background structure.[4] For instance, one usually thinks of inner models like $L(\mathbb{R})$, HOD, etc., as *just* subclasses of V. But as Hamkins himself explains:

> I counter this attitude, however, by pointing out that much of our knowledge of these inner models has actually arisen by considering

[3] For a technical exposition of *forcing*, cf. again the mentioned [Kunen, 2011], or [Jech, 2003].

[4] The reader is warned that 'model' and 'structure' are used throughout interchangeably.

> them inside various outer models. We understand the coquettish nature of HOD, for example, by observing it to embrace an entire forcing extension, where sets have been made definable, before relaxing again in a subsequent extension, where they are no longer definable ([Hamkins, 2012], p. 418)

So, it is not simply through viewing them as *definable* subclasses of V that one comes to understand the very nature of inner models, but rather through studying their behaviour within *other* models, such as, for instance, forcing extensions of another (provisional) background universe V. This is an instance of what I call 'perspectivism' with respect to models: the same models will show different, possibly more interesting, features if studied in the context of (as parts of) other models.

In an effort to further clarify this view, more recently Hamkins has suggested the following characterisation of (PERSP):

> The multiverse perspective ultimately provides what I view as an enlargement of the theory/metatheory distinction. There are not merely two sides of this distinction, the object theory and the metatheory; rather, there is a vast hierarchy of metatheories. Every set-theoretic context, after all, provides in effect a metatheoretic background for the models and theories that exist in that context – a model theory for the models and theories one finds there.
> ([Hamkins, 2020], pp. 297-8)

An equally promising way of construing (PERSP), then, is through envisioning the existence of a *hierarchy* of set theories, each reflecting a (provisional) theory/metatheory distinction with respect to another one, such that: the model theory of a theory T (that is, the 'multiverse' of T) is 'observed' from the point of view of another theory (and provisional *metatheory*) T', the multiverse of T' from the point of view of the theory (and provisional *metametatheory*) T'' and so on. On pain of contradicting (PERSP), there can be no *ultimate* level in this hierarchy, insofar as there is no ultimate background theory from whose point of view one can 'observe' the model theory of other theories.

Now, (PLAT) and (PERSP) may seem to contradict each other, to some extent. After all, platonists take platonic entities to be *well-delineated*, *determinate* and *static* ontological constructs. How is it, then, that they change their 'nature' depending on which model, as postulated by (PERSP), one observes them from? By (PERSP), for instance, a set-theoretic structure, say, $\mathfrak{A}$ could, if observed from the point of view of another structure $\mathfrak{A}'$, look different from the way it could look, if observed from the perspective of another structure $\mathfrak{A}''$.

One major issue relating to HOP, is, as a consequence, its *consistency*, i.e., whether it is able to yield a framework which makes sense of the co-existence between (PLAT) and (PERSP).

What I aim to do in this paper is to describe a metaphysical framework which does not only account for the consistency of HOP, but is also able to provide responses to several concerns about Hamkins' multiverse which have been left unanswered.

2 Higher-Order Platonism: Preliminaries

First, I need to discuss a few further features of HOP, in particular, I should make clear in what sense it could be seen as instantiating *mathematical (set-theoretic) platonism*.

Although accounts of platonism may differ, scholars usually agree on the its basic tenets, which are taken to consist in the following claims:[5]

Existence (E). There *exist* mathematical objects.

Abstractness (A). Mathematical objects are *abstract* (as opposed to *physically instantiated*, *concrete*) objects.

Independence (I). Mathematical objects are *independent* of minds, language, uses and practices.

One further claim is commonly seen as qualifying the position, that is:

Truth-Value Realism (TVR). Mathematical statements are *determinately* true or false.

but note that one may hold TVR without also holding E, A or I.

The E, A, I triad also seems to be part of HOP: that there exist mathematical objects is implicit in Hamkins' 'platonism about universes', only the objects referred to by HOP are not the 'classic', first-order mathematical objects; models of set theory seem to be no less *abstract* than first-order objects, and, finally, 'I' may carry over to HOP without much ado or, in any case, it does not seem to be less appropriate to HOP than it is to classic platonism.

Now, if one takes TVR to be an essential component of classic platonism as much as the E, A, I triad, then HOP noticeably departs from classic platonism, since, as is clear, each set-theoretic model will fix the truth of some set-theoretic statements in a way which might differ from that of another model.[6] Hence, by HOP, it is not always the case that, for a given statement ϕ, ϕ is *true* or *false*: in many cases ϕ will *neither* be true *nor* false (it will be *indeterminate*).

However, one may attempt to rescue TVR by *relativising* it to a specific structure. One could, that is, propose a revised ('extended') version of TVR along the following lines:

[5] For an overview of platonism in mathematics, see [Linnebo, 2018], which I have used here to characterise the fundamentals of the position.

[6] For accounts of platonism which also hold TVR, see, for instance, [Gödel, 1947] and [Martin, 2001]. [Isaacson, 2011], while subscribing to TVR up to the level of ZFC, holds that at least *some* set-theoretic statements (e.g., the existence of large cardinals), are genuinely indeterminate.

ETVR (Extended Truth-Value Realism). Each mathematical (set-theoretic) statement is *determinately* true or false *in* a given mathematical structure.

Moreover, one could also attempt to go beyond the Tarskian concept of 'truth in a model', and define a concept of 'truth *in all* models (of some kind)'. Such an account of truth is, for instance, described by [Woodin, 2011] in connection with the 'set-generic multiverse conception'. The multiverse concept of truth is as follows:

Multiverse Conception of Truth (Falsity). ϕ is true (false) iff it is true (false) in *all* members of the set-generic multiverse, that is, the collection of all set-generic extensions of an initial model M, and of their grounds.[7]

However, it would not seem that HOP-ists should take this road. This is clear if one considers this further feature of Hamkins' multiverse, as expressed in the following quote:

> The background idea of the multiverse, of course, is that there should be a large collection of universes, each a model of (some kind of) set theory. There seems to be no reason to restrict inclusion only to ZFC models, as we can include models of weaker theories ZF, ZF^-, KP, and so on, perhaps even down to second-order number theory, as this is set-theoretic in a sense ([Hamkins, 2012], p. 436).

What can be gleaned from the quote above is that Hamkins' multiverse is not exhausted by any collection of set-theoretic models (of any kind). At this point, this would hardly strike anyone as surprising: already by (PERSP), each universe must 'yield' its own multiverse, and it should not be possible to amalgamate all such multiverses into a unique, coherent model-theoretic superstructure. As a consequence, the Hamkinsian multiversist has to entirely abandon the goal of defining 'multiverse truth' in the way sketched above, although he might, in principle, subscribe to ETVR.

But the quote also provides us with another fundamental piece of information about HOP: *all* models of *all* theories (of sets) are members of Hamkins' multiverse. This means that the Hamkinsian multiversist is not only a *semantic pluralist*, in the sense that he thinks that truth may vary across the multiverse, but also a *proof-theoretic pluralist*, insofar as he thinks that any collection of consistent set-theoretic axioms provides us with a legitimate version of 'set-theoretic truth' (that it meets some 'concept of set').

I will discuss in depth, and fully exploit, the entangled character of the two positions later on; for the time being, it is worth laying emphasis on this further fundamental feature of HOP:

Proof-Theoretic Pluralism (PTP). All consistent theories of sets are legitimate,

[7] This notion is best explicated in terms of satisfaction in Ω-logic, '$\models_\Omega$', for which also see [Woodin, 2001].

i.e., they instantiate equally legitimate versions of set-theoretic truth ('concepts of set').

Our description of HOP is now complete. We have seen that classic platonism's E, A, I triad is kept by HOP, whereas unrelativised semantic determinacy (TVR) is dropped. Moreover, HOP's *semantic pluralism* is also subtly related to *proof-theoretic pluralism*, in a word, the idea that there are as many set-theoretic concepts and versions of truth as formal theories of sets.

3 Higher-Order Platonism: Fundamental Features

I now proceed to address the metaphysics of HOP through focussing on the three main conceptual axes of *plenitude*, *structure* and *reference through description*. In particular, I will focus on the issues of: (1) what is meant by there being a plenitude of higher-order entities, (2) whether these entities can be construed as structures in the sense of some form of mathematical structuralism, and (3) how such entities are accessed through descriptions. I will carry out this task by examining, among other things, alternative accounts of 'plenitudinous platonism' (all predating the emergence of HOP), and by assessing their suitability as metaphysical underpinnings of HOP. This will progressively, but forcefully, lead me to settle on Linsky and Zalta's Object Theory (whose features I will also describe in this section) as the most suitable interpretation of HOP.

3.1 Plenitude

All versions of 'plenitudinous platonism', to which HOP belongs, commit themselves to the view that there is a *plenitude* (as opposed to classic platonism's *scarcity*) of platonic mathematical objects. A way to understand this is through invoking the idea that are as many objects as there *can* be, that the 'logical space', as it were, is completely filled with such objects, and, finally, that conceiving those objects automatically entails conceiving their *totality*.

These ideas are variously combined and expressed in *plenitude principles* asserting the existence of specific *abstracta*, some of which will happen to be *mathematical objects*. By Lewis' *modal realism*, for instance:

> (1) absolutely every way that a world could possibly be is a way that some world is [...] ([Lewis, 1986], p. 86)

i.e., every *conceivable* logical articulation of a world instantiates an *existing* world, so, in the end, there exists a *plenitude* of worlds.

[Balaguer, 1995]'s Full-Blooded Platonism (FBP)'s is paradigmatic:

Full-Blooded Platonism (FBP). Every *conceivable* mathematical object exists.

FBP's notion of 'conceivable' needs, of course, be characterised more formally. Balaguer takes it to be equivalent to: 'logically consistent', so FBP asserts that every consistent mathematical object is an existing (platonic) object. Now, consistent mathematical objects are described by consistent *theories*, thus, by FBP, every consistent theory of sets T describes some platonic objects. Balaguer's example is well-known: ZFC+CH and ZFC $+\neg$CH, despite their manifest overlapping features, describe different platonic objects (that is, 'portions of the set-theoretic realm').

So, HOP could, potentially, be (some form of) FBP. Indeed, one could argue that (PLAT) is met by FBP, since HOP's higher-order entities (models) could be taken to correspond to FBP's 'different portions of the mathematical realm', and since FBP is also plenitudinous with regard to theories, then it also satisfies (PTP).

However, on closer inspection, it is not altogether clear that HOP's (PLAT) is adequately dealt with by FBP. As we have seen, FBP implies that theories describe 'portions of the mathematical realm', but this is a very vague notion, to say the least. In fact, as we know, one theory could, in principle, be satisfied by several 'portions' of the mathematical realm (models). Take, for instance, ZFC. This will have models where CH is true, and models where $\neg$CH is true, models where SH is true and models where $\neg$SH is true, and so on. Which of these opposite properties characterises the 'portion of the mathematical realm' described by ZFC? The FBP-ist will respond that, insofar as ZFC cannot attribute any of these properties to the objects it describes, ZFC is not describing *any* objects having such properties, yet one would need a more systematic account of the relationships among theories.[8]

Moreover, FBP does not seem to be able to express (PERSP): by FBP, there exist *different* universes, but these seem to be rigid, mutually unrelated, constructs.

Lewis' modal realism, on the other hand, seems to violate more basic platonistic requirements, since it holds that a *world* W is (or rather, may be made to correspond to) a region of space-time, so, in particular, W is (also) a *subset* of the *continuum*, and this in violation of 'A', which, as we have seen, takes platonic objects to be *abstract*.[9]

In general, plenitudinous versions of platonism place a lot of emphasis on the *non-spatio-temporality* of abstract objects, as this reinforces (and also justifies) the appeal to *plenitude principles*: since platonic objects do not have *sharp* edges, as they would, if they were spatio-temporally located, they can only be thought of as *totalities*, plenitudes of logical possibilities.

3.2 Structures

The objects referred to by HOP's (PLAT) are, by definition, the models of set theory, and we have seen that these may not fit too well with FBP's universes

[8]However, [Balaguer, 1998], p. 59, explains that theories could be *nested*, in the sense that the 'portion of set-theoretic realm' of one (e.g., ZFC) could be properly included in the 'portion of set-theoretic realm' of another one (ZFC+CH).

[9][Lewis, 1986], p. 118, explains that: 'a world is a distribution of a two-valued magnitude over a continuum of spacetime points'.

('portions of the mathematical realm'), even less so with the modal realist's *worlds*. An alternative, very natural way of thinking of such objects is through an independently given, and platonistically construed, notion of mathematical *structure*.

I will examine two forms of structuralism which are relevant to my purposes: set-theoretic structuralism (STS), and *sui generis* structuralism (SGS).

STS really looks like a good approximation of HOP, since it is precisely the idea that mathematics is about structures conceived as *sets*, in particular, as the *models* of certain theories T. So, by STS, number theory is about the $\mathbb{N}$-structure(s), analysis is about the $\mathbb{R}$-structure(s), etc. Now, is ZFC about the ZFC-structure(s)? According to [Hellman, 2005], p. 540, ZFC strikes a major disanalogy with other theories, since:

> ..on its face value interpretation, set theory itself is *not* treated structurally: its axioms are not understood as defining conditions on structures of interest but are taken as assertions of truths in an absolute sense. One speaks of the "*the* cumulative hierarchy", or even "the real world of sets".

Hellman assumes that set theory, contrary to other mathematical theories, is *non-algebraic*, that is, it is not interpretable as being about alternative set-theoretic structures, but about just *one* structure (the universe of sets V); by this view, set theory represents an exception to the applicability of STS itself. But, clearly, in the context of HOP, one could say that, since set theory is, in fact, not about a *single* universe, but rather about *several* universes, the *models* of set theory, then set theory is no exception to STS, and HOP could, really, be seen as a form of STS.

The weakness of this interpretation lies in another feature of STS (as standardly formulated): structures ought not to be interpreted as *platonic* (higher-order) entities. In particular, STS' structures could be seen, in explicit violation of the platonist's core tenets, as being *eliminated*.[10]

HOP's (PLAT) may more aptly be rendered by a different version of structuralism, SGS, which I proceed to examine.[11] SGS holds that isomorphism types (of structures), that is, *structural invariants*, are independently existing, platonic entities. That is, structures are, by SGS, *ante rem* objects which cannot be eliminated, so it would seem that we are done.

But, again, SGS commits itself, even more than STS, to a single-structure characterisation of set theory, and, in fact, of all mathematical theories for which it provides a repository of *ante rem* structures, such as the *natural number structure*, the *real number structure*, etc. As a consequence, SGS is usually coupled with a second-order logic account of mathematical theories, all of which

[10]In the sense that, by STS, what count as structures are the *models* (that are always *sets*), not the *ante rem* structures (*structural invariants*), of a theory. Cf. [Hellman and Shapiro, 2019], p. 37.

[11]The standard reference for this form of structuralism is [Shapiro, 1997]. Cf. also [Hellman and Shapiro, 2019], p. 51ff. [Hellman, 2005], p. 542, calls SGS, in a way strongly evocative of what I have called here HOP, 'hyperplatonism'.

can thrive on the *categoricity* of their respective axioms. This is, for instance, expressed by Shapiro as follows:

> ..categorical characterisations of the prominent infinite mathematical structures are available. Because isomorphism, among systems, is sufficient for "same structure", a categorical theory characterizes a single structure if it characterizes anything at all. ([Shapiro, 1997], p. 133)

In the case of set theory, such a single structure is $(V_\kappa, \in)$, where κ is the least inaccessible cardinal, the *unique*, up to isomorphism, model of the second-order theory ZFC_2+'there is no inaccessible cardinal'. This fact already nails down the manifest incompatibility of HOP with SGS.

But even if SGS provided scope for a multiple-structure interpretation of set theory, it could not fulfill (PERSP) anyway. By SGS, structural invariants are fixed constructs, each endowed with a fixed isomorphism type τ. But, by (PERSP), models do not need to have such a feature (a model U might have type τ, from the point of view of a model V, and type σ from the point of view of another model W).

Finally, one could also, potentially, turn to a form of topos-theoretic structuralist approach. Since *toposes* are *categories*, this approach would be an instance of *category-theoretic structuralism* (CTS), whereby mathematical structures are *categories*.[12] Recently, [Blechschmidt, 2022], has attempted to apply topos theory to models of set theory, i.e., to the set-theoretic multiverse, but, as the author himself makes clear, there are robust disanalogies between model theory and topos theory, and, in any case, it is not clear that Hamkins' multiverse could be any kind of topos.[13]

3.3 Descriptions

Access (and *reference*) to higher-order objects in the context of plenitudinous platonism are different from access (and reference) to first-order objects in classic platonism.

The 'austere' platonist is, on the one hand, (mostly) construed as believing in a special form of *intuition*, which would provide her with a kind of knowledge of mathematical objects as certain (and determinate) as knowledge of physical objects.[14] Plenitudinous versions of platonism, on the other hand, purport to be able to eliminate the somewhat mysterious character of such an intuition. This is because they hold, as we have seen, that what one really has access to when one refers to abstracta are totalities of them, which are accessible

[12] For a general overview of forms of CTS, see [Hellman and Shapiro, 2019], pp. 43ff., and [McLarty, 2004]. Very roughly, toposes are categories, that is sets of objects and morphisms, very much like the 'category of all sets', which is the category-theoretic rendition of the 'universe of sets' V.

[13] [Blechschmidt, 2022], pp. 70ff. On this, also see [Hamkins, 2012], fn. 2, p. 417.

[14] The classic expression of this view is in [Gödel, 1947], but also see Gödel's further arguments (and explanations) in [Gödel, 1951]. For a useful review of this conception, see [Parsons, 1995].

through some *comprehension principle*, such as FBP, and the latter, in turn, just requires the understanding of basic logical notions, such as that of '*consistency* of a theory'.[15]

Linsky and Zalta, with whose form of plenitudinous platonism, that is, Object Theory (OT), I will be concerned in the next few paragraphs, make clear that knowledge of (and reference to) abstracta are made possible by *logical description*. Among other things, they think, this allows platonists to successfully address the classic Benacerrafian concern about how one should account for knowledge of *non-spatiotemporal* objects.[16] The authors explain that:

> Knowledge of particular abstract objects does not require any causal connection to them, but we know them on a one-to-one basis because de re knowledge of abstracta is by description. All one has to do to become so acquainted de re with an abstract object is to understand its descriptive, defining condition, for the properties that an abstract object encodes are precisely those expressed by their defining conditions. So our cognitive faculty for acquiring knowledge of abstracta is simply the one we use to understand the comprehension principle. ([Linsky and Zalta, 1995], p. 547)

'Encoding', referred to in the quote above, lies at the heart of OT, so it should be explained straight away.[17] This consists in a new form of *predication*, denoted 'xF', which means: 'x encodes F' (as opposed to the well-known 'Fx', which means: 'x exemplifies F'). The difference between exemplification and encoding can be explained as follows.

An object x exemplifies a property P if that property can be *predicated* of x: '2 is prime' is a case in point. An object x encodes a property P if x *is* that property. '2 encodes *primality*' is a case in point. Abstract objects will both exemplify and encode properties. Encoding is regulated by the following Comprehension Principle:[18]

$$(\exists x)(A!x \wedge \forall F(xF \leftrightarrow \phi)) \qquad \text{(Comp)}$$

where $A!x$ means: 'x is abstract', F is a property, ϕ is a set of conditions (expressing the property F). So, (Comp) asserts the existence of a plenitude of abstract objects, each of which encodes all and only those properties expressed by a set of conditions ϕ. In simpler terms, an abstract object is just an object

[15][Balaguer, 1995], pp. 304ff.

[16]Cf. [Benacerraf, 1973].

[17]The material which follows is mostly based on [Linsky and Zalta, 1995] and [Nodelman and Zalta, 2014]. Further details on OT may be found in [Zalta, 2000], and [Linsky and Zalta, 2006]. For an extensive discussion of OT, see [Panza and Sereni, 2013], section 5.2.

[18]The language in which OT is cast is just the language of second-order predicate logic with equality $\mathcal{L}^2_=$, enriched with atomic formulas which express 'encoding'. Sometimes the authors use other special symbols, whose meaning is explained later in the text and fn. 21. Cf. [Zalta, 1983], Ch. 1.

which *encodes* a collection of properties.[19] (Comp) is OT's archetypal Plenitude Principle. It allows the formation of *any kind of* abstract objects, among these, mathematical objects.

But OT is not exhausted by (Comp). The theory extends *abstractness*, and *plenitudinousness*, to all other 'objectifiable' parts of formal languages (and logic), and, among these, crucially, to *theories*.

Formally, a *theory* is just a collection of propositions. In turn, *propositions* are well-formed formulae, consisting of symbols for constants, predicates, relations, all of which are distinct abstract objects. So, theories, qua abstract objects, are combinations of other abstract objects. More specifically, through the corresponding Comprehension Principles, one first forms a plenitude of *relations (predicates)*, then a plenitude of *propositions*, construed as 0-place relations. Finally, one forms a plenitude of theories T by identifying these with T's own *theorems*, that is, with the closure set of the '$\vdash$' relationship, or, *semantically*, with the set of the *logical consequences* of T.[20] The resulting Comprehension Principle is as follows:

$$T \stackrel{\text{def}}{=} \iota x(A!x \wedge (\forall F)(xF \leftrightarrow \exists p(T \models p \wedge F[\lambda y\, p]))) \qquad (\text{Comp}_T)$$

that is, a theory T is the *only* abstract object which encodes all and only those properties asserted by its *true* propositions.[21]

(Comp$_T$) may also be seen as a Comprehension Principle for *structures* (that is, for *models* of a theory T), taken to be the *referents* of 'true' theories. Thus, in practice, OT identifies theories with the *structures* which satisfy them. PA is, for instance, the collection of all propositions true of PA-*structures*, ZFC the collection of all propositions true of ZFC-structures, and so on. Also structures can be referred back to theories via (Comp$_T$). For instance, a model of ZFC which satisfies the CH will be interpreted as the abstract object corresponding to the theory ZFC+CH.

Now, while, like FBP, OT identifies models with *particular theories*, unlike FBP, it also provides a careful explanation of why ZFC is, for instance, different from ZFC+CH: because the abstract object ZFC+CH *encodes* properties which are not encoded by the abstract object ZFC.

OT also envisages the existence of a plenitude of first-order mathematical objects. OT's treatment of such a notion follows in a fully consequential manner from the theory's presuppositions. Let κ be a mathematical object: the exact reference of κ is fixed by some theory T. Thus, there can be no object κ without an accompanying theory T (which fixes its reference). So, κ splits into a multitude of κ_T, one for each theory T. In formal terms:

[19] Alternatively, for any non-empty collection of properties, there is an abstract object which encodes them.

[20] The switch between syntax and semantics is fundamental for Linsky and Zalta's purposes. This can be accounted for, in OT, through using what the authors call Importation Rule, described in [Nodelman and Zalta, 2014], p. 48.

[21] The ι-operator means: 'the only', $[\lambda y\, p]$ is the λ-denotation of the proposition p (the object y such that p), and $F[\lambda y\, p]$ denotes: 'the proposition p which asserts F'.

$$\kappa_T \stackrel{\text{def}}{=} \iota y(A!y \wedge (\forall F)(yF \leftrightarrow T \models F(\kappa_T))) \qquad (\text{Comp}_{obj})$$

(Comp), (Comp$_T$) and (Comp$_{obj}$) are sufficient for my purposes. In fact, only (Comp$_T$) shall prove necessary.

One last remark is in order. *Encoding* also differs from *exemplification*, insofar as abstract objects are 'encoding-incomplete' whilst being 'exemplification-complete'. This means that, say, a natural number κ_{PA}, *encodes* all and only those properties that PA asserts κ to have. Therefore, it may be the case that κ_{PA} does not encode some property F, or its negation. On the other hand, for all F's, either κ_{PA} exemplifies it or its negation. Thus, the encoding-related features of abstract objects constitute a painstaking explanation of *set-theoretic indeterminacy*. Reference to objects is fixed by theories, and these are, by their nature, 'encoding-incomplete', so also first-order mathematical objects (*sets*) will be encoding-incomplete. Hence, theories (of sets) are about inherently *incomplete* objects, which are, however, liable to becoming 'more complete'. A classic example is the real continuum. The object $\mathcal{P}(\omega)_{\mathsf{ZFC}}$ is encoding-incomplete with respect to CH. But this doesn't mean that $\mathcal{P}(\omega)_T$ will, in general, be encoding-incomplete for any T. In fact, $\mathcal{P}(\omega)_{\mathsf{ZFC}+V=L}$ is encoding-complete with respect to CH.

4 Higher-Order Platonism Inside Object Theory

HOP can be adequately interpreted ('embedded') inside OT. This is the present work's main claim, which I will now proceed to discuss. But first I should clarify what I really mean by this claim.

My goal is not to suggest that Hamkins' multiverse is formally reducible (whatever that could mean) to a fragment of OT, nor do I posit that the Hamkinsian multiversist should think that his conception is just OT. What I actually think it that OT provides us with a detailed and coherent metaphysical account of HOP, which enriches our understanding of Hamkins' multiverse in a way that no other form of plenitudinous platonism does.

In order to attain my goal, I will have to show that OT expresses the three main features of HOP, that is, (PLAT), (PERSP) and (PTP). This is a trivial task as far (PLAT) and (PTP) are concerned, but (PERSP) will need an extended examination.

As far as (PLAT) is concerned: (Comp$_T$) envisages the existence of a plenitude of higher-order abstract objects, that is, *theories*. Moreover, as we have seen, (Comp$_T$) also entitles us to construing theories as structures satisfying them, and *vice versa*. It is, in my view, a noticeable strength of OT, as compared to other versions of plenitudinous platonism, that it is also able to incorporate, and refer to, models of set theory. To this end, as we have seen, one just has to pick out the relevant abstract objects corresponding to theories.

Clearly, (PTP) is also met by OT, since any theory of sets represents a self-standing abstract object. Again, it is a defining feature of OT that theories are taken to be platonic objects. All of them are legitimate since OT's Comprehension Principles do not discriminate between 'legitimate' and 'illegitimate' *entities*. Moreover, that any abstract object corresponding to a theory of sets T reflects a 'different concept of set', as envisaged by Hamkins, is expressed, in OT, by the fact that theories *encode* different properties (in particular, they encode different theorems and truths about different structures). So, there is a clear and robust sense in which OT clearly asserts the existence of different 'concepts of set'.

Finally, I proceed to address (PERSP), which, as I have said in section 1, is crucial to HOP's purposes. (PERSP) has been formulated as the view that the multiverse is always viewed from the point of view of (is relative to) a specific model. In order to exemplify the position, I also mentioned Hamkins' own example of *inner models*' being best understood from the point of view of *forcing extensions*.

The other aspect of (PERSP) I have highlighted, and which is also to be reconsidered in light of OT, is that, within Hamkins' multiverse, one is supposed to constantly *jump* from one model to another, and this is also construed (more broadly) by Hamkins as a 'an enlargement of the theory/metatheory distinction'.

In order to illustrate the first aspect, let us go back to the example of inner models inside forcing extensions of a provisional background universe V. One starts with constructing a forcing extension $V[G]$ (for our, and Hamkins' purposes, this could just be any countable transitive model of ZFC). The corresponding Hamkins multiverse, by (PERSP), is, thus, the collection of all models accessible from $V[G]$. Among these, one 'finds' inner models such as L, HOD, $L[\mathbb{R}]$, etc. Any of these will now reveal, inside $V[G]$, features which, in turn, are contingent on the features of $V[G]$ itself.

Now, what kind of abstract objects and properties are, by OT, involved in observing, say, the constructible universe L from the point of view of $V[G]$? The first object under consideration is $V[G]$ itself, that is, a model of ZFC. Now, recall that, by (Comp_T), a model of ZFC is the *same abstract object* as ZFC. So, in particular, any property encoded by $V[G]$ will be *encoded* by the abstract object ZFC. But $V[G]$, presumably, encodes other properties which are not encoded by ZFC, for instance, $\neg$CH (presumably, that was also the reason why we picked out $V[G]$ in the first place). In that case, we should, more correctly, refer $V[G]$ back to the abstract object (and theory) ZFC+$\neg$CH.

But now we need to express, in OT, that ZFC+$\neg$CH does not only encode the features of $V[G]$, but also those of the L *inside* $V[G]$. So, our next object of scrutiny is the sentence:

(Φ). $V[G]$ is a model of ZFC+$\neg$CH, and, moreover, $V[G] \models$ 'L has some property Ψ'.

where 'Ψ' expresses some feature of L as seen from the point of view of $V[G]$.[22]

[22][Hamkins, 2012]'s example, on p. 418, is Cohen's construction of a $V[G]$ wherein $L \models \neg$AC.

Now, Φ is a metatheoretic statement about models of ZFC+$\neg$CH, so we cannot expect that it is really expressible in ZFC+$\neg$CH. But, given the representability of ZFC+$\neg$CH's metatheory in ZFC+$\neg$CH itself, we may find, in the object theory, an equivalent statement, say, $\Phi_{\mathsf{ZFC}+\neg\mathsf{CH}}$, that, somehow, expresses Φ. Now, $\Phi_{\mathsf{ZFC}+\neg\mathsf{CH}}$ is a *bona fide* abstract object. In particular, by (Comp$_T$), $\Phi_{\mathsf{ZFC}+\neg\mathsf{CH}}$ is a truth about some structure satisfying ZFC+$\neg$CH itself, and, thus, can, finally, be viewed as being encoded by ZFC+$\neg$CH. In the end, in OT, one has that the abstract object ZFC+$\neg$CH encodes a property which expresses a 'property of L *inside* a model $V[G]$ of ZFC+$\neg$CH', and this is precisely what was required of me to establish.

In order to discuss the second aspect of (PERSP), i.e., the idea of 'jumping' from one model to another, let us carry on with the example of L inside $V[G]$.

I seem to live in L (in fact, the L of $V[G]$), so my Hamkins multiverse consists of all models accessible from L itself. On OT's conception, this just means that now I am reasoning from the point of view of a different theory, that is, ZFC+V=L, so, again, through (Comp$_T$), I am picking out a different abstract object, corresponding to the theory ZFC+V=L. In other terms, the jump from $V[G]$ to L can now be expressed as the shift from one abstract object to another, which is, in turn, tantamount to picking out (describing) *distinct* abstract objects. If we wished to jump from there to another model M, then we will just have to pick out the abstract object corresponding to the theory whose theorems are *true* in M and, of course, this process can be iterated as many times as one pleases.

One further technical remark is necessary. Of course we cannot expect ZFC+$\neg$CH (or ZFC, for that matter) to prove that it has models (like $V[G]$) which satisfy $\neg$CH and further properties (of their L, for instance). The theory will, at most, prove, that some arbitrarily finite fragment of that theory has a *countable* model with the mentioned properties, so the truth encoded by the abstract object corresponding to ZFC+$\neg$CH will refer to such models. This is not ideal, but it is inevitable. On the other hand, if one used Hamkins' *toy model perspective*, whereby all models are countable from the beginning, then one would get that the multiverse, and all relevant metamathematical facts about it, would be encoded only by the abstract object ZFC.[23] But this approach would be awkward, for then, in particular, the jump from a model to another would just be a jump from one countable model of ZFC to another countable model of ZFC. Through keeping the principle of identifying a structure with the corresponding theory, we can, on the contrary, modulo the mentioned limitations, construe the jump from a model to another one as picking out *distinct* theories (abstract objects), something which seems to be a lot more faithful to the motivation behind Hamkins' multiverse.

To sum up, OT is able to interpret (PLAT), (PERSP) and (PTP), which, overall, means that OT is able to interpret HOP. Let me empasise, again, that, in particular, OT is flexible enough to interpret the jumps from a model to another one which are so characteristic of Hamkins' multiverse. The crux of OT's

[23]For a discussion of the toy model perspective, see [Hamkins, 2012], p. 436ff.

intepretation of (PERSP) consists in transforming metamathematical facts, on which the Hamkinsian multiversist thrives, into *properties* encoded by abstract objects corresponding to the relevant theories.

5 Three Problems: Articulation, Skepticism, Practice

Let's take stock. I have scrutinised several forms of higher-order platonism (some of which plenitudinous), and concluded on a positive note about HOP's interpretability inside OT. I have also made it clear that OT should be taken to be a 'companion' theory to, not a formalisation of, HOP.

Now, HOP (and Hamkins' multiverse, for that matter) have sparked much controversy and raised concerns relating both to the underlying logic and the surrounding philosophy. In what follows, I will show that some of these concerns may be successfully addressed if one adopts the conception and tools afforded to us by OT.

5.1 Articulation

[Koellner, 2013] raises a problem of 'articulation' for what he calls the 'broad multiverse conception', that is, Hamkins' conception.[24] The problem may be explained as follows.

The existence of a set-theoretic multiverse, that is, of all models of some theory of sets T, is secured through assuming the consistency of T. But, by Gödel's Incompleteness Theorem, one cannot hope to prove Con(T) in T, so, in order to get Con(T), one needs to use a stronger theory as a background theory. Now, suppose one has ascended to a stronger theory $T' = T + Con(T)$ and can, now, articulate the multiverse of T. This would clearly be insufficient for the Hamkinsian multiversist, as he still cannot refer to models of T', so one will have to ascend, again, to a stronger theory, $T'' = T' + Con(T + Con(T))$ to articulate the multiverse of T', and so on. Overall, this will result in an infinite regress through background theories of increasing consistency strength, none of which exhausts the 'whole' Hamkinsian multiverse.[25]

The response to Koellner's argument is, in the light of OT, straightforward. The HOP-ist does not need to articulate the 'whole multiverse' in the way suggested by Koellner. In fact, by (PERSP), he doesn't even have a fixed concept of 'set-theoretic multiverse'. It is true that he needs to start from some theory T, in order to build an initial multiverse, but, by (Comp$_T$), members of such mul-

[24] In fact, [Koellner, 2013], pp. 4-5, distinguishes between the 'broad multiverse conception', and the 'relative broad multiverse conception', that is, the narrowing of the former to just one background theory T. For my purposes, it will just suffice to deal with the 'broad multiverse conception'.

[25] [Koellner, 2013], pp. 7-8. A similar argument pointing out, this time, a 'referential regress' in Hamkins' multiverse is in [Barton, 2016], p. 202.

tiverse will come to the fore automatically, since they are, so to speak, inbuilt features of the abstract object corresponding to the theory itself.

So, what the HOP-ist needs to do to articulate *some* multiverse is just to pick out the abstract object corresponding to his initial theory T, and the properties this encodes. Among those properties, there will be the relevant metamathematical facts about the models of T. Afterwards, she can either pick out another theory (and abstract object) T or, by (PERSP), she can progressively explore all other models accessible from the initial multiverse.

Moreover, the HOP-ist does not need to take into account the issue of the consistency of T, insofar as OT does not discriminate between consistent and inconsistent abstract objects. In particular, some abstract objects will encode inconsistent theories. However, by OT, this does not prevent such theories from existing: only, there are *no* abstract objects which *exemplify* them, so in particular there are *no* structures which exemplify them.[26]

[Koellner, 2013] also lays emphasis on what he thinks are further paradoxical features of Hamkins' multiverse relating to consistency.[27] Given any theory T (of sufficient consistency strength), Hamkins' multiverse will also contain models of $T + \neg Con(T)$. Now, in order to secure the existence of those models in the multiverse, one has to presuppose that T is, in fact, *consistent*, so there is a tension between the 'internal' and 'external' claims of consistency of T. But, again, my interpretation of HOP gets rid of the distinction between internal and external. Access to abstract objects corresponding to theories is carried out through description, so no *external* presupposition need to be made concerning the properties encoded by those objects, in particular, as we have seen, concerning their consistency. The existence of a (non-standard) model of $T + \neg Con(T)$ entirely lies 'inside', and is encoded by, the abstract object $T + \neg Con(T)$, so the issue of the clash between the external and internal claims of consistency of such theory, practically, vanishes.

5.2 Skepticism

Does HOP instantiate a form of *skepticism* about set theory as well as mathematical knowledge at large?

In order to answer this question, one should, presumably, have at hand a clear notion of what would count as 'skepticism', and the issue is too broad to even start examining it here. However, a few considerations on this issue may be inevitable, especially insofar as the charge has been levelled many times by different authors.[28]

One straightforward way of arguing that HOP is a form of skepticism is through raising the issue of *referential indeterminacy* in connection with it. The

[26] [Linsky and Zalta, 1995], p. 537, fn. 32, mentions the famous example of the 'round square': by OT, there exists a unique abstract object encoding these two *mutually inconsistent* properties, but there is *no* object which exemplifies them.

[27] Cf. [Koellner, 2013], pp. 8ff.

[28] Cf. [Koellner, 2013], pp. 9. [Button and Walsh, 2018], pp. 205ff., take Hamkins' multiversism to be a contemporary version of Skolem's 'model-theoretic skepticism', for which cf. [Skolem, 1967].

argument, broadly, runs as follows: set-theorists seem to have *determinate* (*unique*) referents in mind for set-theoretic objects, whereas HOP subscribes to (Comp$_{obj}$), which implies, among other things, that referents of set-theoretic objects are *non-unique* in an essential way; hence HOP cannot represent an accurate interpretation of the set-theoretic discourse, let alone a foundation for it.

One possible response is that this argument may be thriving on a (subtle) confusion between 'relativism' and 'indeterminacy'. Granted, HOP is relativistic, since properties of set-theoretic objects are deemed by it to be contingent on theories (and structures modelling them). Yet, it would not be correct to view HOP as supporting the referential indeterminacy of the set-theoretic discourse. HOP (and OT, for that matter), do not hold that the reference of, say, 1 is *indeterminate*; rather, they hold that it is *plural*. According to HOP-ists, and OT-ists, 1 is an incomplete denotation of the object 1_T, that is, of 1 as defined within some theory T. Now, they may agree that, when mathematicians (and set-theorists) refer to 1, they refer, most of the times, to 1_{PA}, but, will, as is clear, deny that this is inevitable. For instance, 1_{ZF} is a sensible alternative that comes with a lot of new theory. One could insist that the interpretation of the arithmetical discourse is so plain and evident, that there is no reason to assume that the referent of a natural number is contingent on a specific theory, but I think that this only begs the question.

HOP-ists' appeal to the relativity of the set-theoretic discourse should be more correctly construed as expressing the belief in the *multiple realisability*, not in the *indeterminacy*, of the discourse itself. Hamkins has stressed this point many times:

> [...] let me remark that this multiverse vision, in contrast to the universe view with which we began this article, fosters an attitude that what set theory is about is the exploration of the extensive range of set-theoretic possibilities. ([Hamkins, 2012], p. 440)

As regards the issue of whether such a view instantiates any form of skepticism, he explains that:

> ..to my way of thinking, this label is misapplied, for the multiverse position is not especially or necessarily skeptical about set-theoretic realism. We do not describe a geometer who works freely sometimes in Euclidean and sometimes in non-Euclidean geometry as a geometry skeptic. ([Hamkins, 2020], p. 295-6)

Now, it seems to me that my interpretation of HOP *within* OT adds crucial weight to this argument. Believers in the existence of all conceivable abstract objects do not deny the *existence* (or *knowability*) of (mathematical) reality. They think, instead, that mathematical reality consists of a very rich realm of realit*ies*, from which one can pick one's preferred mathematical concepts. As we have seen, in OT, this idea has prominently featured as the claim that mathematical objects are *non-spatiotemporal* and *sparse*.

The accusation frequently levelled against *plenitudinous platonism* of being a brand of formalism, that is, of 'anti-realism', is not new in the debate, in any case, and discussing it in full is besides the scope of this paper.[29] Presumably, whether one may have a coherent concept of 'multiple (mathematical) realities' also depends on what one takes 'reality' to consist, and on how the latter (causally or non-causally) interacts with human minds. On such purely epistemological grounds, which I cannot examine here, some people will keep insisting that there is a *unique* mathematical reality and, therefore, that plenitudinous platonism (as well as HOP) have a skeptical attitude about set-theoretic reality.[30] But this is a claim we ought not to light-heartedly grant.

5.3 Practice

Our theorising, so far, has been 'metaphysically thick'. By HOP, in order to understand what set-theorists really do, one does not only need to understand set theory mathematically, but also understand some accompanying, explanatory theory of set-theoretic objects, theories and structures like OT.

Thus, one further objection against HOP would be that one doesn't actually need to do this. That is, one could say that, whatever the merits of the position, regardless of whether it's coherent or not, of whether it is sound or not, such an extreme form of set-theoretic realism really has no impact on set-theorists' work. Moreover, they would continue, the multiversist seems to want to make things more complicated than they are, since taking onboard a complex, platonistically construed, multiverse ontology affords no potential gains in foundational terms.

Naturalists, such as Maddy, have expressed this point of view. In particular, Maddy has stressed that Universism *and* the theory ZFC+Large Cardinals are more than sufficient for all our mathematical and philosophical purposes, where the role of 'philosophy' is taken, by Maddy, to consist in that of a Second Philosophy, that is, of a philosophy which attends to the 'details of practice' and does not aim to reinterpret or influence in any way practitioners' work.[31]

[Maddy, 2017] even laments the potential harmfulness of metaphysical incursions into mathematical work thus, overall, questioning the usefulness of HOP:

> [t]he metaphysics of abstracta or meanings or concepts are all really beside the point. The fundamental challenge these multiverse positions raise for the universe advocate is this: are there good reasons to pursue a single, preferred theory of sets that's as decisive as possible, or are there not? ([Maddy, 2017], p. 316)

[29]For instance, [Field, 2001], pp. 334ff. takes FBP to be of an 'anti-objectivistic' character, and [Potter, 2004], p. 11, even denies that FBP has anything to do with platonism. [Linnebo, 2018], p. 21, on the other hand, views such conceptions as FBP and OT as belonging 'somewhere in the territory between anti-nominalism and full-fledged platonism'.

[30]Cf., for instance, [Martin, 2001].

[31]For the articulation of Second Philosophy, cf. [Maddy, 2007].

We already know Maddy's answer to her own question: the intra-theoretic reasons to pursue just one theory, ZFC+Large Cardinals and 'single-V', are very cogent, since this theory best expresses the foundational goals of set theory.

Now, although the universist orthodoxy, so to speak, is far from being overthrown, several works have appeared which put pressure on this conception. For instance, [Ternullo, 2019] has suggested that the goals mentioned by Maddy might also be attained by a multiverse theory, and that further goals, such as that of studying the relationships between models (or theories), *qua* constitutive of the 'multiverse undertaking', clearly suit best multiverse theories.[32] Moreover, 'multiverse maths' has delivered many promising mathematical programmes, and many more might appear in the next future.[33]

More recently, [Antos, 2022] has further pointed out that also a Second Philosopher could see models of set theory as being part of set theory's 'fundamental entities'. This is because:

> [i]ntroducing models as fundamental entities is part of set-theoretic methodology and not just a heuristic aid. Being about methodology, this claim then becomes eligible to inform further philosophical questions [...]. ([Antos, 2022], p. 6)

Of course, at this point, one could observe that one thing is to hold that the set-theoretic multiverse (or some conception thereof) might fit with a purely naturalistic methodology, and another is to advocate a complex, set-theoretically dispensable, metaphysical theory such as OT.

In other terms, while the naturalist could, in principle, be open to embrace [Mostowski, 1967]'s formalistic pluralism about 'set theory', as expressed in this quote:

> Probably we shall have in the future essentially different intuitive notions of sets just as we have different notions of space, and will base our discussions of sets on axioms which correspond to the kind of sets which we want to study. ([Mostowski, 1967], p. 94)

she could not possibly subscribe to the metaphysical theorising inherent in HOP. At which point, the HOP-ist could reply that HOP is consistent with the kind of naturalism that also includes Mostowski's pluralism and Antos' suggestion that models are indispensable mathematical entities. As a consequence, the anti-platonist naturalist will now turn to launch a full-scale attack against HOP's noticeably platonistic doctrines, and the HOP-ist will respond, as Hamkins does, that platonism is 'natural', insofar as the *experience* of models of set theory is genuine, and cannot be explained away.[34] Then, the naturalist will respond that the HOP-ist's notion of 'experience' is not hers, and so on. This debate will, most likely, end in a stalemate.

[32] Cf. [Ternullo, 2019], pp. 57ff.

[33] One can immediately think of one of those, i.e., set-theoretic 'geology', for which see [Fuchs et al., 2015], now thought to be central to many set-theoretic undertakings.

[34] [Hamkins, 2012], especially p. 418.

It might be too early in any case to assess whether HOP's foundation of set theory is going to fully convince the naturalistically-minded philosophers and set-theorists (incidentally, this might also depend on the status of Universism, as well as of Universism-inspired mathematical programmes, in the coming years).

What is certain is that HOP seems to have noticeable epistemological advantages over other conceptions: correct or not, it provides an *explanation* of why set theory is so incomplete, by pointing out that set-theoretic concepts are *incomplete*, in particular, via OT, *encoding-incomplete*, i.e., they do not have enough 'content' to be able to provide unique solutions to fundamental set-theoretic problems. It might not be too much for a naturalist, but clearly having at hand some sort of principled, 'meta-set-theoretic' explanation of incompleteness will sound appealing to many.

6 Concluding Remarks

I have shown that there is a way to make sense of 'higher-order realism' in the context of Hamkins' multiverse, by (informally) embedding what I have called HOP into a general theory of abstract objects, OT. Such an embedding does not require of one to see HOP as a 'fragment' of OT, only to acknowledge that HOP's platonistic conception can be adequately expressed by OT.

In particular, the Hamkinsian multiversist's typical statements that *models* are set-theorists' main objects of investigation, that we have a genuine experience of them, that set-theorists jump from one universe to another, become fully intelligible in the context of OT. Moreover, as we have seen, OT comes with an explanation of set-theoretic incompleteness, which, in my view, makes the 'multiverse undertaking' even more persuasive. Of course, one can resist the force of the explanation, but, certainly, the latter stands Hamkins' multiversism in good stead.

Finally, I have also shown that, through OT's reinterpretation of HOP, the Hamkinsian multiversist may also be able to respond to issues concerning the articulation of the multiverse, and reject as, at least in part, inappropriate, the label of 'skepticism' for his conception.

OT presents itself as an elegant (and economic) solution to the interpretative problems posed by (PLAT), (PERSP) and (PTP). The basic objects are taken to be theories, and these are reduced to the structures which satisfy them. So the complex metatheory of the multiverse which is needed to express salient facts about the models of set theory can be incorporated, through the very flexible and apt notion of *encoding*, into the theories themselves.

There are other accounts of higher-order platonism out there in the market, but none of these, at least, none of those scrutinised in this work, seems to be as suitable as OT. This does not mean that other suitable platonistic accounts of models of set theory, of multiversism at large, are likely not to be found.[35]

[35] For instance, although not directly addressing set-theoretic pluralism, [Horsten, 2019]'s meta-

On the contrary, it is now even more legitimate to ask:

Question 1 *In which ways could the study of the metaphysics of HOP be further developed, and with what foundational implications?*

Once shown that it is concretely possible to bring the metaphysical features of the multiverse to bear on the foundations of set theory, where does all this leave us? Does such incursion into metaphysics imply some major revision of our accepted and practiced methodologies in set theory?

My answer has been, and still is, a 'no', especially insofar as HOP does not directly bear on the mathematical methodologies in any way. But I think that the potential usefulness of 'auxiliary conceptions' in the context of the foundations of mathematics should not be denied straight away. This is because, as Hamkins himself asserts:

> ..our foundational ideas in the mathematics and philosophy of set theory outstrip our formalism, mathematical issues become philosophical, and set theory increasingly finds itself in need of philosophical assistance. ([Hamkins, 2012], p. 436)

although, admittedly, the dividing line between 'philosophical assistance' and 'philosophical reinterpretation of mathematics' might, potentially, be very hard to draw.

This last remark very aptly leads me to pose one further question:

Question 2 *To what extent should extra-set-theoretic, metaphysical, in particular, resources be taken to bear on set-theoretic work?*

At this stage, I could not say anything on this issue which would not look like as just a collection of superficial remarks. I will just limit myself to observing that the present work may, potentially, contribute to the view that, once the extra-set-theoretic resources are properly formalised and adequately interpreted, they might really become part of (even explain) the underlying logic and concepts of set theory in a meaningful way, and not just play the role of a 'useful heuristic', in the Maddian sense. But clearly it would still be a long way to go to make a fully convincing case for this claim.

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physical account of 'arbitrary objects' seems very promising in this respect. For a comparison between Horsten's approach and OT, see section 8.3 of that work.

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