Cantor's Abstractionism and Hume's Principle

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December 10, 2020

Two Problems/1.

Hume's Principle (HP)

Given any two *concepts* F and G, the *number* of Fs is equal to the *number* of Gs if and only if the things falling under F are in a one-to-one correspondence with the things falling under G.

Cantor's Principle (CP)

Given any two *sets* P and Q, the *power* of P is equal to the *power* of Q if and only if there is a one-to-one corrspondence between the elements of P and the elements of Q.

- Both HP and CP define (cardinal) numbers by *abstraction*
- Both HP and CP lie at the roots of a theory of infinite cardinalities
- *Problem*: Frege criticised Cantor's *abstractionist* strategy behind CP, and much subsequent literature has also found Cantor's strategy unsatisfactory or unconvincing. But are the two authors' strategies really *irreconcilable*?

Goal 1 of the Paper

A suitable reconstruction of Cantor's abstractionism makes the latter *quite compatible with* Frege's conceptions.

Two Problems/2.

Cantor's (Whole-Part) Principle (CP)

Given two sets A and B, $A \subset B \Longrightarrow \mathfrak{s}(A) \leq \mathfrak{s}(B)$.

Aristotle's Principle (AP)

Given two sets A and B, $A \subset B \Longrightarrow \mathfrak{s}(A) < \mathfrak{s}(B)$.

- CP and AP (may) express different *sizes* for the same infinite sets.
- AP is motivated by an *abstraction principle* of the same force as that motivating CP (as well as HP)
- Therefore, authors have argued that AP is as justified as CP (and HP)
- *Problem.* Is the *kind of abstraction* behind AP the same as that behind CP (and HP)?

Goal 2 of the paper (Cantorian Argument for HP)

CP Show that CP (and HP) instantiate a *stronger and more plausible* form of abstraction.

Goal 3 of the paper

Use the strategy adopted for Goal 2 to tackle other Good Company Arguments against the HP.

Contrasting Assessments

Set theory, as we know it, resulted from a confluence of two distinct historical developments, one beginning from the work of Gottlob Frege from the 1870s to the early 1900s, the other beginning from the work of Georg Cantor during roughly the same period. That Frege's initial motivations were at least partly philosophical, while Cantor's were at first largely mathematical, only serves to highlight the rich conceptual roots from which the theory arose. ([Maddy, 1997], p. 3)

Set theory is that branch of mathematics whose task is to investigate mathematically the fundamental notions 'number', 'order', and 'function', taking them in their pristine, simple form, and to develop thereby the logical foundations of all of arithmetic and analysis ([Zermelo, 1908], p. 200)

Frege's approach to mathematics was substantially different from Cantor's, and though Cantor could appreciate and even praise Frege's aims, he could never bring himself to adopt the austerity of Frege's program. More than anything else, Frege believed that the principles upon which arithmetic must be founded were essentially logical in character. (in [Dauben, 1979], 220)

But, natural or not, it is a mistake to see Cantor's remarks as anything like a nod in the direction of the Frege-Russell theory. For one thing it will emerge later that Cantor's conception of the Absolute rules out the 'whole class' approach to number, as Cantor himself was probably aware. But for another this association ignores Cantor's own account of abstraction. And this leads us in a quite different direction. ([Hallett, 1984], p. 125)

Zermelo's Assessment

> *...what Frege understands by 'number [Anzahl]' is exactly what Cantor denotes by 'cardinal number' [Kardinalzahl], namely the invariant, which is common to all equivalent (Frege says 'equinumerous') sets (Frege says 'concepts'). It is just that Frege identifies the class-invariant with the 'extension of the concept: equinumerous with the concept F. [...] For us today it can only seem striking and regrettable that the two contemporaries, the great mathematician and the commendable logician, have, as this review shows, understood each other so little (Zermelo 1932, transl. in [Ebert and Rossberg, 2009, 347]).*

In the passage above, Zermelo claims the fundamental *indistinguishability* of Cantor's and Frege's conceptions (at least up to [Cantor, 1885b]).

The literature has, on the contrary:

1. placed emphasis on the differences ([Dummett, 1991], [Hallett, 1984], [Dauben, 1979])
2. trashed or re-interpreted Cantor's abstractionist account of number ([Tait, 1997], [Fine, 1998])
3. by following Frege's own assessment of Cantor's strategy (in [Frege, 1892]) mostly proposed a Frege *vs.* Cantor picture ([Ortiz Hill, 2004], [Tait, 1997])

In the paper, we wish to do justice to Zermelo's view.

What Is an Abstraction Principle?

Abstraction principles may be defined as follows as *universally quantified biconditionals* of this form:

$$\forall\alpha\forall\beta(\Sigma(\alpha) = \Sigma(\beta) \leftrightarrow \alpha \sim \beta) \qquad \text{(Abs)}$$

where α and β are variables, 'Σ' is a term-forming operator (e.g. 'the number of') and $\sim$ stands for an *equivalence relation* over entities of that type.

For instance, Σ could be: 'the direction of a (straight) line (Dir)', α, β could be 'lines' and '$\sim$' the relational predicate 'parallel to'. In this case, we say that (Dir) is defined *by abstraction* over parallel lines.

Fact

Both *HP* and *CP* are abstraction principles.

Proof. They are specific instances of (Abs). E.g.,

$$\forall F\forall G\ (\#(F) = \#(G) \leftrightarrow F \approx G) \qquad \text{(HP)}$$

(However, note that HP quantifies over *second-order* entities (*concepts*), CP over *first-order* entities (*sets*)).

Cantor's First Conception

It may be found in: [Cantor, 1879], [Cantor, 1883], and [Cantor, 1885a].

Cantor's First Definition of Number (C_1)

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A cardinal number is the *general concept* under which fall all and only those sets which are *equipollent* to a given set.

C_1 (Equinumerosity)

$$card(x) = card(y) \iff x \sim y (\iff \exists f : x \to y \textit{ such that } f \textit{ is } 1-1 \textit{ and onto})$$

C_1 (Definition of Number)

$$card(x) =_{df} \#x = \{y : y \sim x\}$$

where the '$\sim$' is the *equinumerosity relation*, and $\#x$ is the *general concept* obtained through abstracting over *equinumerous sets*.

Cantor's Second Conception

We will call by the name "power" or "cardinal number" of M the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given. We denote the result of this double act of abstraction, the cardinal number or power of M, by $\overline{\overline{M}}$. Since every single element m, if we abstract from its nature, becomes a "unit", the cardinal number $\overline{\overline{M}}$ is a definite aggregate composed of units, and this number has existence in our mind as an intellectual image or projection of the given aggregate M. ([Cantor, 1895], p. 86)

Every ordered aggregate M has a definite "ordinal type" or more shortly a definite "type" which we will denote by $\overline{M}$. By this we understand the general concept which results from M if we only abstract from the nature of the elements m, and retain the order of precedence among them. Thus the ordinal type $\overline{M}$ is itself an ordered aggregate whose elements are units which have the same order of precedence amongst one another as the corresponding elements of M, from which they are derived by abstraction. ([Cantor, 1895], pp. 111-2)

Cantor's Second Conception/2.

The second conception features in: [Cantor, 1888], [Cantor, 1890], and, most prominently, in [Cantor, 1895]. We may summarise [Cantor 1895]'s formulation as follows:

C_2 (Ordinal Numbers)

For all sets M, *ordinal* numbers are *ordinal sets* $\overline{M}$, resulting from abstracting over all other properties of $m \in M$, except for the *order*.

C_2 (Cardinal numbers)

For all sets M, *cardinal* numbers are *cardinal sets* $\overline{\overline{M}}$, resulting from abstracting over *all* other properties of $m \in M$.

In Cantor's view, the abstraction process, then, turns $m \in M$ into 'units'.

The Definition of Number in Frege's *Grundlagen*

In §§62-3 of [Frege, 1884], Frege suggests that numerical identities of the form 'the number of F = the number of G' can be defined contextually by means of HP.

This provides Frege with an indirect (*abstractionist*) definition of *number*.

Further remarks:

- Even though Frege dismisses HP in §67 (because of the Caesar Problem), the principle remains central to his foundational project. HP can indeed be derived from Frege's explicit definition, as Frege himself notices.
- Frege nowhere provides an *argument* for HP, in spite of this principle's centrality in the development of his theory. Indeed, Frege did provide support for HP *vis-á-vis* other forms of axiomatic definitions.
- At the same time, Frege provides no justification for taking one-to-one correspondence to be the *correct* standard for the equality of numbers.
- Although this is speculative, it is very likely that Frege may have seen the use of HP as fully justified *on the grounds of* the parallel use of CP by Cantor (cf. [Mancosu, 2016], ch. 3)

Frege's 1892 review of Cantor's [Cantor, 1890]

Frege produced two versions (of which the former was never to be published) of a review of [Cantor, 1890]. The latter also contained [Cantor, 1888], where Cantor's C_2 may be found.

Frege advanced two main objections to C_2, both of which he traces back to his *Grundlagen*:

- Cardinal numbers should not be abscribed to *sets* or *pluralities* of objects, but to the *concept* under which the members of those pluralities fall (irrelevant?)
- Second, he argued that the process of *abstraction* as progressive removal of an object's identifying properties is impossible.

On this, he says:

> *if we try to produce the number by putting together different distinct objects, the result is an agglomeration in which the objects contained remain still in possession of precisely those properties which serve to distinguish them from one another; and that is not the number. But if we try to do it in the other way, by putting together identicals, the result runs perpetually together into one and we never reach a plurality* [1].

[1] [Frege, 1884, 50].

A summary of (Frege's and others') objections to C_2

- *Persistence*. Units will preserve their *individuality*, if *collections* thereof are to be produced.
- *Arbitrariness*. Units will be chosen arbitrarily: for instance, we may say that $1 = \{u\}$ and $2 = \{v, w\}$. Now, why should we have that $u \in 1$ rather than $u \in 2$?
- *Indiscernibility*. While units should be distinct entities, they will, in fact, become indistinguishable (as, ultimately, all units are the *same unit*).
- *Psychologism*. C_2 refers to 'acts of abstraction', to cardinal sets as 'projections' and 'images' of the original set.

Three Interpretations of C_2

- Tait's *Discontinuity Thesis*. Tait has held that the shift from C_1 to C_2 should be viewed as a clear (and unfortunate) rupture in the evolution of Cantor's use of abstraction.

  It would be interesting to conjecture about the reasons for the change between the point of view of (1883) to that of (1887-88); but I shall not go into that here, other than to register my regret ([Tait, 1997], in [Tait, 2005], p. 216).

- Fine's *Ontological Thesis*. Fine, in [Fine, 1998] shows how one can defend C_2 based on a specific ontological interpretation of Cantor's 'units':

  Abstraction, as Cantor and Dedekind conceive of it, is ontologically innovative – it leads to objects which are genuinely new. [...] As I hope will become clear, however, it is only by taking ontology seriously that we can come to a satisfactory view of what these authors had in mind. ([Fine, 1998], p. 601)

- Hallett's *Ordinal Thesis*. Hallett (in [Hallett, 1984]) suggests that 'units' are just (are comparable to) ordinals. So, cardinals are just collections of ordinals.

The Explanation Thesis

Our interpretation of C_2 hangs on the following thesis:

The Explanation Thesis

C_2 is an explanation of C_1, which means: C_2 is an explanation of how C_1 works 'practically'.

Our thesis implies that C_2 has two fundamental meanings:

1. One covers the *function* (of units): units are just 'labels' for elements of a set M in the range of the *pairing function* in C_1.
2. One covers the principle of *substitution invariance* (for units): C_2 expresses the idea that objects (elements in any set M) may be replaced with each other. According to this interpretation, 'unit=any object (of thought)'.

For the second part, consider the following remark by Cantor, who, at some point, in [Cantor, 1895], explains:

> *[A]ccording to the above definition of power [that is, C_2, our note], the cardinal number $\overline{\overline{M}}$ remains unaltered if in the place of each of one or many or even all elements m of M other things are substituted. ([Cantor, 1895], p. 88)*

The Explanation Thesis and the Objections to C_2

- *Persistence*, *Arbitrariness* and *Indiscernibility* are successfully dealt with by the Explanation Thesis: units *are not* objects, so they should not be either uniquely individuated or mutually distinguished.
- *Psychologism* is not addressed, as 'units' are taken to act as a convenient *metaphor* of what happens when one pairs elements of two *sets*.
- Main upshot: C_2 and C_1 do not contain *different* conceptions of abstraction.

Gödel's Minimal Account of Abstraction

At the very beginning of his [Gödel, 1947], Gödel says:

> *For whatever 'number' as applied to infinite sets may mean, we certainly want it to have the property that the number of objects belonging to some class does not change if, leaving the objects the same, one changes in any way whatsoever their properties or mutual relations (e.g., their colors or their distribution in space). From this, however, it follows at once that two sets (at least two sets of changeable objects of the space-time world) will have the same cardinal number if their elements can be brought into a one-to-one correspondence, which is Cantor's definition of equality between numbers. For if there exists such a correspondence for two sets A and B it is possible (at least theoretically) to change the properties and relations of each element of A into those of the corresponding element of B, whereby A is transformed into a set completely indistinguishable from B, hence of the same cardinal number ([Gödel, 1947], p. 254)*

Cont'd.

Gödel's quote lays emphasis on precisely the two aspects ('meanings') we lay emphasis on in our Explanation Thesis: *function* and *substitutional invariance*. Gödel's analysis leads to the following:

Gödel's Minimal Account of Abstraction (GMAA)

For an abstractionist definition of number to work, two conditions need to be met:

1. There has to be a pairing function (correspondence) among any two *items (entities)*
2. The nature of the entities involved should not matter

We believe that **GMAA** provides necessary (but not sufficient) conditions for the Fregean, necessary and sufficient conditions for the Cantorian, to separate *good* from *bad* abstraction.

Fact

Both HP and CP satisfy the **GMAA**.

Therefore, although HP and CP are not *as* justified for the Fregean, they are, approximately, *almost as* justified, insofar as they both satisfy **GMAA**: we claim that this is already a lot.

Persisting issues: (1) psychologism (2) irreducibility of concepts to sets.

Mancosu's Representative-Based Account of C_2

An alternative to our Explanation Thesis is Mancosu's conception of definitions by abstraction as hinged on 'representatives'. Mancosu says:

> *[T]hese qualms could be overcome only with the development of a theory of cardinals and ordinals in which the choice of representatives for these numbers could be effected by means of the axiom of choice which allowed a development of the theory of equivalence with canonical representatives. In this way the abstraction principle for cardinals and ordinals moves from a postulation of abstract objects to a choice of representative sets for cardinals and ordinals. ([Mancosu, 2016], p. 52)*

Mancosu's Representative-Based Interpretation of C_2

C_2 identifies 'representatives' within classes defined through C_1, which are none else but the *cardinals* of Cantor's *cumulative number-class hierarchy*.

Positive features of Mancosu's account:

1. It does justice to the *objectual* component of the idea of a 'cardinal set'
2. Cantor was not too happy about classes in C_1 (?)
3. Cardinals in the number-class structure may be construed as *platonic abstracta*

Cont'd.

The cumulative number-class structure (the sequence of all *transfinite cardinalities*) defined by Cantor in [Cantor, 1895] is as follows:

$$\aleph_0 = \{\alpha : card(\alpha) < \aleph_0\}, \aleph_1 = \{\alpha : card(\alpha) \leq \aleph_0\}, \aleph_2 = \{\alpha : card(\alpha) \leq \aleph_1\}, \cdots$$

where α ranges over *Ord*.

Ultimately, Mancosu's *representatives* might just be the 'standard' transfinite cardinals appearing in the *cumulative number-class structure*.

The main problem with this interpretation is that it is entirely hinged on C_1 insofar as, in it, C_2 does not play any role, so this might just be a re-statement of either Tait's *Discontinuity Thesis* and Hallett's *Ordinal Thesis*.

Moreover, it is not clear that 'representatives' would fit the Fregean's conception of *abstraction*.

Heck's Challenge to the Inevitability of HP in [Heck, 1997]

Heck's Challenge. The possibility of a different assignment of cardinal numbers to infinite concepts would show that HP is not *analytic*, since it is possible that another principle, instead of HP, underlies our actual concept of cardinal number.

In Heck's own words:

> *...if it is conceptually possible that infinite cardinals do not obey HP, then it is conceptually possible that HP is false, which means that HP is not a conceptual truth, so HP is not implicit in ordinary arithmetical thought*[2].

Heck's Challenge requires the availability of:

1. Alternative notions of *cardinality*
2. A proof that such notions do not obey HP
3. Most importantly, in order to assuage the (neo-)Fregean's potential concerns, the proof that they obey some other abstraction principle of the *same force as* HP

Heck shows that alternative notions of *cardinality* are available, all of which satisfy 1-3.

[2] [Heck, 1997, 2001 Postscript, p. 641]

Alternative Notions of Cardinality

[Mancosu, 2016] shows, however, that this objection – the 'Good Company' problem – can be replicated using principles which share all the 'good' features that neo-Fregeans ascribe to HP.

Mancosu also shows ([Mancosu, 2016], pp. 170 ff.), indeed, that there are (infinitely many) principles similar to HP, but which differ from it on the assignment of cardinal numbers to infinite concepts.

- *Peano's Principle* (PP) assigns then the *same cardinal number* to every pair of infinite concepts, no matter whether those concepts are equinumerous with each other.
- *Boolos' Principle* (BP) assigns (i) the same cardinal number a to each concept F which is *both infinite and co-infinite* (*viz.* such that the extension of $\neg F$ is also infinite); (ii) a different cardinal number b to each concept G which is *infinite but co-finite* (*viz.*, such that the extension of $\neg G$ is finite); and (iii) cardinal numbers to finite concepts in the same way as HP.

All of these principles, as shown by Mancosu, are 'good', in the sense that they satisfy the neo-Fregean's requirements that they may be formulated as second-order statements.

However, do they satisfy Cantor's notion of a *good abstraction* as encapsulated by **GMAA**?

Numerosities

One further, more recent, and interesting case in point is the theory of numerosities formulated by Vieri Benci, Mauro Di Nasso and Marco Forti (cf. [Benci and Di Nasso, 2003], [Benci et al., 2006], [Benci and Forti, 2020]). The theory supports AP:

Aristotle's Principle (AP). If $A \subset B$, then $\mathfrak{s}(A) < \mathfrak{s}(B)$.

Note that AP is, in turn, none but the mathematical incarnation of what has often been called Euclid's Axiom (EA):

Euclid's Axiom (Common Notion)

The whole is greater than the parts.

The three Italian mathematicians have recently developed a theory of numerosities (**TN**) alternative to Cantor's theory, which:

1. Is based on a new way of counting
2. Produces a model of non-standard analysis (numerosities are *hyperfinite* numbers)
3. Introduces an *arguably* maximal notion of *real-closed (non-Archimedean) ordered field* isomorphic to the *ordered field of the surreals* (the *absolute continuum*), and the Ω-*saturated* field of the *hyperreals*, that is, the *Euclidean continuum* $\mathbb{E}$ (cf. [Benci and Forti, 2020]).

TN's Challenge

Moreover, numerosities may also be defined by abstraction using the equivalence relation given by a special *ultrafilter* $\mathcal{F}$ on the natural numbers:

$$num(A) = num(B) \leftrightarrow A \sim_{\mathcal{F}} B \qquad \text{(NUM)}$$

(NUM) is an *abstraction principle* for numerosities.

TN's Challenge: the theory of numerosities represents a viable (and better) alternative to Cantor's set theory based on a *good* kind of abstraction.

In order to assess the challenge, we need to examine a few further details of the theory.

Numerosities: counting and labelling

Intuitively, numerosities are based on a different way of 'counting'.
There are two main ways to count:

1. Through using one-to-one correspondences (CP)
2. Through using *partial sums* (**TN**)

Examples

Suppose A is a (denumerably) infinite collection of boxes: $A_0, A_1, \ldots, A_n, A_{n+1}, \ldots$, each of which contains an object, except for A_0, which contains 10 objects. Then, by CP, the number of objects is ω, whereas by **TN**, it is $10 + \omega$. If I take out an object from each *even-indexed box*, then, by **TN**, I get $\omega/2$, by CP, I always get ω, and so on.

So, counting depends on *grouping*, which, in **TN**, is called 'labelling'.

Definition

A labelling is a pair $< A, \ell(A) >$, where A is a set (the domain) and $\ell(A)$ is a function from A to $\mathbb{N}$, which assigns a number (the *label*) to each object in A.

Using labelling, we can produce *partitions* of a set A, so that $A = A_0 + A_1 + A_2 + \ldots$, where the elements of each A_n are entirely specified by the labelling function $\ell(A)$.

Now, labelling may vary. **TN** has mostly taken into account the *canonical labelling*: $\ell(\alpha) = \alpha$ for all ordinals α (in particular, if A is a subset of $\mathbb{N}$, $\ell(n) = n$).

Numerosities: approximating sequence and the ultrafilter

Consider:

$$S = s_0 + s_1 + s_2 \ldots + s_n + s_{n+1} + \ldots$$

where each term is the size of each A_n for a labelling $\ell(A)$. This is called the 'approximating sequence' for A: $\gamma_A : n \to S_n$.

Definition (Numerosity)

$num(A) = [\gamma_A]$

Thus, we have that numerosities may be construed as particular strictly increasing sequences $< s_n >$ of natural numbers.

Definition (Numerosity function)

Given the class of labelled sets $\mathcal{L}$, a numerosity function is the the function $num : \mathcal{L} \to \mathcal{N}$, which assigns a unique numerosity to each labelled set.

Finally, $\mathcal{N}$ is defined as follows:

$$\mathcal{N} = \mathbb{N}^{\mathbb{N}}_{\mathcal{F}} = \{[\phi]_{\mathcal{F}} \mid \phi : \mathbb{N} \to \mathbb{N}\}$$

where ϕ is an approximating sequence.

Note: the existence of $\mathcal{F}$ is independent of ZFC! (Martin's Axiom is needed to prove its existence).

Resisting **TN**'s Challenge

We are now ready to assess **TN**'s challenge.

- The notion of a *definition by abstraction*, as encapsulated by **GMAA** presupposes:
  1. The use of a pairing function (correspondence)
  2. That properties of the entities involved do not count.

Consider, again, (NUM):

$$num(A) = num(B) \leftrightarrow A \sim_{\mathcal{F}} B$$

Now, (NUM) violates both requrements of **GMAA**:

1. $A \sim_{\mathcal{F}} B$ requires the definition of an ultrafilter on $\mathbb{N}$
2. Properties of the entities involved count: e.g., $\mathbb{N}_{odd} \subset \mathbb{N}$, thus, by AP, $\mathfrak{s}(\mathbb{N}_{odd}) \neq \mathfrak{s}(\mathbb{N})$.

(One further issue is labelling: there are different ways of making the labelling, which yield different way of producing *numerosities*).

Claim

(NUM) does not satisfy **GMAA**, therefore it is not a *good* abstraction principle (or a suitable alternative to CP).

Responses to Heck's Challenge

The strategy to tackle the Good Company Arguments for all other alternative cardinal notions is similar.

For instance:

- As far as PP is concerned: CP supports a *uniform* way of assigning cardinal numbers to concepts, since the process of cardinal abstraction is the same in the infinite case as in the finite ones. PP, by contrast, treats finite and infinite concepts differently: thus it violates point (2.) of **GMAA**.
- As far as BP is concerned: Recall that BP assigns the same cardinal number to all infinite but co-finite concepts, and a different number to all infinite and co-infinite concepts. Again, in order to apply this criterion, one cannot proceed, in general, by abstraction from the F-things in the way outlined by Cantor. By contrast, one would need to consider the things that do *not* follow under F, namely the complement of the set to which cardinality is assigned. Again, point (2.) of **GMAA** is violated.

Summary of the Cantorian Argument for HP

- Cantor's conception of a *good* definition of number *by abstraction* is encapsulated by **GMAA**.
- The latter requires that: (1) one uses a pairing function between items (concepts, sets); (2) properties of such items do not count
- HP satisfies **GMAA**
- Thus, HP is sanctioned by the Cantorian conception
- HP has been challenged by several other abstraction principles
- If alternatives to HP are equally good, then HP is not inevitable
- Alternatives to HP do not satisfy **GMAA**
- Cantor's conception, therefore, leads to reject all alternatives to HP
- By the Cantorian conception, HP is still inevitable

The End

Thanks for your attention!

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