Cantor's Abstractionism and Hume's Principle

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In both principles '$\cong$' means 'is equinumerous to', that is: 'there is a bijection between the Fs and the Gs/the members of X and the members of Y'.

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- These authors have pointed out that the possibility of defining *cardinal numbers* in ways which violate HP (and CP) puts a lot of pressure on the neologicist strategy.

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- Clearly, PWP contradicts CP (and HP), and the theory of numerosities (see next few slides) obeys PWP.

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- PP, BP and NUM can all be formulated as *second-order abstraction principles*. In Mancosu's view, this shows that HP has serious contenders, which may be no less compelling and intuitive than HP itself.

Hale on Heck's and Mancosu's Arguments

Some have thought that if one can coherently take different countably infinite sets to differ in size, it follows that Hume's Principle cannot be a conceptual truth. But that inference seems far too swift. It assumes, not only that the theory of numerosities captures a concept of cardinal number, but that it seeks to capture a single concept of cardinality of which the Cantorian conception, enshrined in Hume's Principle, offers a rival account. Mancosu is more cautious—the theory of numerosities and the Cantorian theory are 'not in conflict. Conflict emerges only if both notions are taken to explicate the same intuitive notion of size', [...]. He does, however, view the former as offering an alternative conception of cardinality to Cantor's, [...]. While much more needs to be said than I can say here, it seems to me that Gödel was right on this point. Indeed, getting different numerosities for different countable infinite collections depends not only upon grouping their members in certain ways, but also on taking the groups in a certain order—but it is of the essence of cardinal as opposed to ordinal numbers that the cardinal size of a collection does not depend upon how its members are ordered. ([Hale, 2018], p. 162)

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- Show that careful analysis of the notion of 'abstraction' leads to see HP's rivals as not being on a par with HP.
- Same analysis, on the other hand, leads to fully support CP and, thus, also HP.
- So, neologicists may use this as evidence that HP is not on an equal footing as its 'good companions', and, moreover, that HP enjoys a special status among abstraction principles.

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Given a set M, its cardinal set, $\overline{\overline{M}}$, is a special set which originates from a *double act of abstraction*, from the *nature* and *order* of the elements of a set M. By abstraction, the elements m of M are turned into 'units' [*Einheiten*], which are the members of $\overline{\overline{M}}$.

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- [Hallett, 1984]: first account is incomplete; second one needs the theory of *ordinals*.

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    2. *Substitutional Invariance*: Any element of M may be transformed into any other element of N
- Thus, in our view, the second account just plays an *explanatory* role, and is *inseparable* from the first one.

Gödel on Cantor's Cardinality Conception

For whatever 'number' as applied to infinite sets may mean, we certainly want it to have the property that the number of objects belonging to some class does not change if, leaving the objects the same, one changes in any way whatsoever their properties or mutual relations (e.g., their colors or their distribution in space). From this, however, it follows at once that two sets (at least two sets of changeable objects of the space-time world) will have the same cardinal number if their elements can be brought into a one-to-one correspondence, which is Cantor's definition of equality between numbers. For if there exists such a correspondence for two sets A and B it is possible (at least theoretically) to change the properties and relations of each element of A into those of the corresponding element of B, whereby A is transformed into a set completely indistinguishable from B, hence of the same cardinal number ([Gödel, 1947], p. 254).

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- As a consequence, both PP and BP do not comply with the criteria expressed by GMAA.

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Set of Numerosities

The set of all numerosities is defined as: $\mathcal{N} = \{[\phi]_{\mathcal{U}} \mid \phi : \mathbb{N} \to \mathbb{N}\}$, where ϕ is an approximating function.

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- Moreover, in order to define $\mathcal{N}$, one has to assume a theory stronger than ZFC.
- So, NUM thus has strong anti-abstractionist features.

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- Cantor’s abstractionist account of number should be interpreted as fixing the criteria for ‘good abstraction’. This is Gödel’s interpretation which we have expressed through GMAA.
- All HP’s rivals violate GMAA, so, HP’s rivals are not on a par with HP as far as abstraction is concerned.
- The neologicist now has a ‘Cantorian argument’ at hand: she might still have to deal with the objections to Cantor’s abstractionist account, but she has means to respond to such objections.

End Slide

Thanks for your attention!

Benci, V. and Di Nasso, M. (2003).
Numerosities of labelled sets: a new way of counting.
Advances in Mathematics, 21:505–67.

Benci, V., Di Nasso, M., and Forti, M. (2006).
An Aristotelian notion of size.
Annals of Pure and Applied Logic, 143(43-53).

Cantor, G. (1885).
Review of Frege's *Die Grundlagen der Arithmetik*.
Deutsche Litteraturzeitung, 6(20).

Cantor, G. (1895).
Beiträge zur Begründung der transfiniten Mengenlehre, 1.
Mathematische Annalen, 46:481–512.

Fine, K. (1998).
Cantorian Abstraction: a Reconstruction and Defense.
Journal of Philosophy, 95(12).

Frege, G. (1884).

The Foundations of Arithmetic.
Pearson-Longman, New York.

Frege, G. (1892a).
Entwurf zu einer Besprechung von Cantors Gesammelten Abhandlungen zur Lehre vom Transfiniten.
Unpublished.

Frege, G. (1892b).
Rezension von G. Cantor: Zur Lehre vom Transfiniten.
Gesammelte Abhandlungen aus der Zeitschrift für Philosophie und philosophische Kritik, 100.

Gödel, K. (1947).
What is Cantor's Continuum Problem?
American Mathematical Monthly, 54:515–525.

Hale, B. (2018).
Paolo mancosu: Abstraction and infinity.
Journal of Philosophy, 115(3):158–166.

Hale, B. and Wright, C. (2001).

The Reason's Proper Study: Essays Towards a Neo-Fregean Philosophy of Mathematics.
Oxford: Clarendon Press.

Hallett, M. (1984).
Cantorian Set Theory and Limitation of Size.
Clarendon Press, Oxford.

Heck, R. (1997).
Finitude and Hume's Principle.
Journal of Philosophical Logic, 26(589-617).

Mancosu, P. (2016).
Abstraction and Infinity.
Oxford University Press, Oxford.

Parker, M. (2013).
Set Size and the Part-Whole Principle.
The Review of Symbolic Logic, 6(4):589–612.

Tait, W. (1997).

Frege versus Cantor and Dedekind: on the Concept of Number.
In Tait, W., editor, *Frege, Russell, Wittgenstein. Essays in honor of Leonard Linsky*, pages 213–248. Open Court, Chicago.