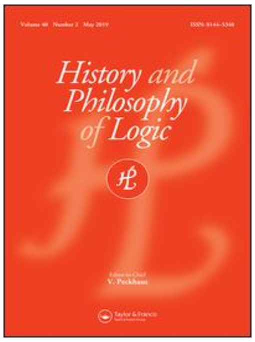

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Cantor's Abstractionism and Hume's Principle

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Richard Kimberly Heck and Paolo Mancosu have claimed that the possibility of non-Cantorian assignments of cardinalities to infinite concepts shows that Hume's Principle (HP) is not implicit in the concept of cardinal number. Neologicism would therefore be threatened by the 'good company' HP is kept by such alternative assignments. In his review of Mancosu's book, Bob Hale argues, however, that 'getting different numerosities for different countable infinite collections depends on taking the groups in a certain order – but it is of the essence of cardinal numbers that the cardinal size of a collection does not depend upon how its members are ordered'. This paper's goal is to implement Hale's response to the Good Company problem by producing a Cantorian argument for HP. In Section 2, we present the Heck-Mancosu argument against neologicism. In Section 3, we discuss Hale's defence of Hume's Principle. In Section 4, we discuss Cantor's abstractionist definitions of number. In Section 5, we argue that good abstraction must comply with what we call 'Gödel's Minimal Account of Abstraction' (GMAA). We finally show (Sections 5 and 6) that non-Cantorian theories of cardinality fail to satisfy GMAA.

1. Introduction

In the final sections of *Grundlagen* Frege hinted at a surprising mathematical result: all the standard axioms of second-order Peano Arithmetic (PA^2) can be derived from Hume's Principle (HP), which states that the cardinal number of F is identical to the cardinal number of G if and only if F and G can be paired one-to-one.[1] Following Frege's suggestion, 'neologicist' philosophers of mathematics, for example *Hale and Wright 2001*, have claimed that HP provides an implicit (or 'contextual') definition of the meaning of the cardinality operator 'the number of'. Frege's theorem[2] would hence show that the axioms of arithmetic are *analytic* in Frege's sense, that is, that those axioms can be derived from (second-order) logic and (an implicit) definition alone.

Richard Kimberly Heck and Paolo Mancosu have argued, however, that the possibility of non-Cantorian assignments of cardinalities to infinite concepts in a way that preserves part-whole intuitions, in particular, the Part-Whole Principle (PWP) – which asserts that a *proper* part of a collection must be *smaller* than the collection itself – entails that HP is not implicit in our ordinary notion of cardinal number (see in particular the Appendix to *Heck 1997* in *Heck 2011* and *Mancosu 2016*, Ch. 4). Neologicism would therefore be threatened by the 'good company' HP is kept by such alternative cardinality assignments. In response to that, in his review of Mancosu's book, Bob Hale has noticed that

> getting different numerosities for different countable infinite collections depends not only upon grouping their members in certain ways, but also on taking the groups

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[1] *Boolos 1990* calls it *Frege's Theorem*. A sketch of the proof was provided by *Wright 1983*, pp. 154–69 based on *Frege 1974*, §§70–83; as noticed by *Boolos and Heck 1998*, however, 'Frege [. . .] cannot be said to have outlined, or even to have intended, any correct proof [of the successor axiom]', and so of Frege's Theorem.

[2] Cf. fn. 1 above.

> in a certain order – but it is of the essence of cardinal as opposed to ordinal numbers that the cardinal size of a collection does not depend upon how its members are ordered.

This paper's goal is to implement Hale's response to the Good Company problem using a 'Cantorian' argument. In Section 2, we present the Heck-Mancosu argument against neologicism. In Section 3 we discuss Hale's defence. In Section 4, we present and discuss Cantor's *abstractionist* definitions of number and argue that they are mutually connected. In Section 5, we argue that good abstraction must comply with what we call 'Gödel's Minimal Account of Abstraction' (GMAA). We finally show (Sections 5 and 6) that non-Cantorian theories of cardinality fail to satisfy GMAA.

2. Neologicism and the Heck-Mancosu Argument (The Good Company problem)

HP is stated as the second-order universal closure of the following formula:

$$\#x : F(x) = \#x : G(x) \leftrightarrow F \approx G \tag{HP}$$

where F and G are monadic second-order variables that range over concepts,[3] '$\#x : F(x)$' is a unary second-level term-forming operator 'the number of', which denotes a function from concepts to objects and, finally, '$F \approx G$' abbreviates a second-order formula that asserts that the F's and the G's can be put into one-to-one correspondence:[4]

$$\exists R(\forall x(F(x) \rightarrow \exists !y(G(y) \wedge R(x,y))) \wedge \forall x(G(x) \rightarrow \exists !y(F(y) \wedge R(y,x)))) \tag{$\approx$}$$

Adopting Frege's terminology, we say that F and G are *equinumerous* if $F \approx G$.

Now, Heck claims, on the one hand, that (1) in order for neologicism to be successful, 'it must be possible to recognize the truth of HP by reflecting on fundamental features of arithmetical reasoning' (*Heck 1997*, p. 597) – that is, HP, including the criterion for assigning cardinalities to infinite concepts that it embodies, must be implicit in ordinary arithmetical reasoning.[5]

Heck points out, however, that Frege Arithmetic (FA), namely, the theory of second-order logic with HP as its sole non-logical axiom, is strictly stronger than Finite Frege Arithmetic (FAF), where HP is substituted by *Finite Hume* (HPF), which states that if either F or G are finite, then F and G have the same number if and only if F and G are equinumerous:[6]

$$\mathrm{Fin}(F) \vee \mathrm{Fin}(G) \rightarrow [\#x : F(x) = \#x : G(x) \leftrightarrow F \approx G] \tag{HPF}$$

HPF can be seen as a restriction of the range of application of HP to finite concepts: indeed, HPF assigns cardinal numbers to finite concepts in the same way as HP, but it makes no claim about the cardinality of the infinite ones. FAF is, in its turn, strictly stronger than PA^2. Heck argues that

[3] A concept, in the Fregean sense, is an 'incomplete' entity that is the referent of predicative expressions; see *Hale 2013* for the neo-Fregean interpretation of second-order logic.

[4] '$\exists !x\, \phi(x)$' is defined as: $\exists x(\phi(x) \wedge \forall y(\phi(y) \rightarrow x = y))$.

[5] See *Macbride 2000* for a critical assessment of Heck's claim with respect to the philosophical aims of neologicism, and see *Heck 2011*, §1.3 for a response; note also that by 'arithmetic' Heck means the arithmetic of finite cardinals.

[6] '$\mathrm{Fin}(F)$' in the formula below abbreviates one of the equivalent definitions of Dedekind-finitude in second-order logic; cf. e.g, *Shapiro 1991*, pp. 100–1 and *Heck 1997*, pp. 589–91. Note that the notion of 'finite' is defined by Frege in terms of the inductive cardinals and, thus, of the notion of cardinal number. So, if $\mathrm{Fin}(F)$ were replaced by 'Frege finite', HPF would be circular. Note also that one can prove that the notion of 'Dedekind infiniteness' implies the notion of 'Frege infiniteness', but the proof of the converse requires the axiom of choice. We are grateful to an anonymous refere for drawing our attention to this issue.

> recognizing the truth of HPF does not require making the conceptual advance made by Cantor; [. . .] Just as HP may be thought of as the sole axiom of a general theory of cardinal numbers, HPF may be thought of as the sole axiom of a theory of finite cardinals. And since HPF makes no claims whatsoever about the conditions under which infinite concepts have the same cardinality, it will not give rise to any of the antinomies generated by HP, whence one does not need to make Cantor's leap before one can accept the truth of HPF.[7]

Since HPF is sufficient to derive PA^2, they conclude, the neologicists have no reason to regard HP rather than HPF as implicit in the ordinary notion of number.

But, what is more important for our purposes, Heck also claims that (2) the possibility of coherent assignments of cardinalities to infinite concepts in a way that preserves 'part-whole intuitions' entails that 'it is conceptually possible that HP is false, which means that HP is not a conceptual truth' (*Heck 2011*, p. 641) – that is, that HP does not underlie our concept of cardinal number.

(2) is hinged on reflections on the history of mathematics. Moreover, in recent times a new way of assigning cardinalities to infinite sets which respects PWP has been developed in the context of what is known as the *theory of numerosities* (**TN**). In compliance with PWP, but also further elaborating upon it, **TN** entails that the *parts* of an infinite set are never as *dense* as the set itself, an idea which is clearly in conflict with HP (and, as we shall see later, also with Cantor's Principle). Instead, two infinite sets have the same size if and only if the functions which, respectively, express their density are equal to each other in a very specific sense, which we shall discuss in full in Section 6. Such functions are called *numerosities*. Functions which 'behave in the same way' belong to the same equivalence class (modulo an *ultrafilter*), so one can express the equality of *numerosities* in the following way:

$$num(A) = num(B) \leftrightarrow A \sim_{\mathcal{U}} B \qquad \text{(NUM)}$$

where A and B are sets, $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and $\sim_{\mathcal{U}}$ is an equivalence relation which we will rigorously define in Section 6.

Now, Heck's point is not that the ordinary concept of number is inconsistent, insofar as it would conflate HP and PWP; Heck's point is rather that the ordinary notion of number 'does not commit itself when both F and G are infinite' and so that neither HP nor the PWP are implicit in that notion. However, if ordinary arithmetical thought is neither committed to HP nor to PWP, then, clearly, it is not committed to HP, contrary to what the neologicists (should) argue.[8]

Mancosu 2016 pushes (2) – Heck's suggestion that HP does not underlie the concept of cardinal number on the grounds that it would not be the only available *cardinality principle* – even further. Mancosu starts by showing that it is possible to formulate infinitely many principles that are similar to HP but differ from it on the assignment of cardinal numbers to infinite concepts. One of Mancosu's examples is *Peano's Principle* (PP):[9]

$$\#x : F(x) = \#x : G(x) \leftrightarrow [(\neg \mathrm{Fin}(F) \wedge \neg \mathrm{Fin}(G)) \vee (\mathrm{Fin}(F) \wedge \mathrm{Fin}(G) \wedge F \approx G)], \qquad \text{(PP)}$$

Unlike HP, PP assigns the same cardinal number to all infinite concepts, regardless of whether or not those concepts can be put into one-to-one correspondence. Then Mancosu

7 *Heck 1997*, p. 599.

8 Note that neologicists themselves have resisted the suggestion that their project rests on the assumption that HP is implicit in ordinary arithmetical thought; see the references in fn. 5.

9 PP captures the behaviour of Peano's *num* function; see *Mancosu 2016*, p. 171.

considers *Boolos' Principle* (BP):[10]

$$\begin{aligned}\#x : F(x) = \#x : G(x) \leftrightarrow (&[\neg\text{Fin}(F) \wedge \neg\text{Fin}(G) \wedge \neg\text{Cof}(F) \wedge \neg\text{Cof}(G)]\\ &\vee [\neg\text{Fin}(F) \wedge \neg\text{Fin}(G) \wedge \text{Cof}(F) \wedge \text{Cof}(G)]\\ &\vee [\text{Fin}(F) \wedge \text{Fin}(G) \wedge F \approx G]), \qquad \text{(BP)}\end{aligned}$$

'Cof(F)' is defined as: Fin($\neg F$). BP therefore assigns the same cardinal number to each infinite and co-infinite concept F – that is, such that both F and $\neg F$ are infinite – and a different cardinal number to each concept that is infinite but co-finite. Finally, Mancosu generalises BP to each natural number n as follows:

$$\begin{aligned}\#x : F(x) = \#x : G(x) \leftrightarrow (&[\neg\text{Fin}(F) \wedge \neg\text{Fin}(G) \wedge \neg\text{Cof}(F) \wedge \neg\text{Cof}(G)]\\ &\vee [\neg\text{Fin}(F) \wedge \neg\text{Fin}(G) \wedge \text{Cof}(F) \wedge \text{Cof}(G) \wedge S_n(\neg F) \wedge S_n(\neg G)]\\ &\vee [\neg\text{Fin}(F) \wedge \neg\text{Fin}(G) \wedge \text{Cof}(F) \wedge \text{Cof}(G) \wedge \neg S_n(\neg F) \wedge \neg S_n(\neg G)]\\ &\vee [\text{Fin}(F) \wedge \text{Fin}(G) \wedge F \approx G] \qquad \text{(BP-}n\text{)}\end{aligned}$$

S_n abbreviates a second-order predicate which is true of a concept F if and only if $\neg F$ has cardinality n. For each n, BP-n assigns the same cardinal number to all infinite and co-infinite concepts, a different cardinal numbers to all infinite and co-finite concepts whose co-finality is n, and a third cardinal number to all infinite and co-finite concepts whose co-finality is different from n.

Mancosu highlights that each of those principles is inconsistent with HP on the assumption that the same cardinality operator figures in both.[11] If we limit ourselves to countable sets, BP entails that the set of the natural numbers $\mathbb{N}$, and the set $\mathbb{N} - \{1\}$ have different cardinalities, whereas HP entails that their cardinality is the same. A contradiction ensues from the supposition that 'Hume's numbers' – that is, the cardinal numbers individuated by HP – are identical to 'Boolos' numbers', and more precisely that the HP-number of F is identical to the BP-number of F for any concept F.

Now, Mancosu's argument runs as follows: (1) PP, BP, and the infinite instances of BP-n ('BP-ns') have the same form as HP, namely that of a second-order universally quantified biconditional stating that two concepts have the same abstract just in case those concepts stand in a given equivalence relation (*abstraction principles*). (2) Like HP, all those principles are *purely logical*, in the sense discussed by *Fine* 2002, that is, their right-hand side can be formulated using only the second-order logical vocabulary.[12] (3) In addition, PP, BP, and the BP-ns are all consistent, and they all comply with the formal criteria that the neologicists have formulated in order to distinguish HP from 'bad' principles such as Frege's inconsistent Basic Law V (this is the reason why, to use the neologicist terminology, HP's rivals are all 'good').[13] (4) Finally, each of those principles is sufficient to derive PA^2.[14]

In light of (3) and (4) above, the 'Good Company' problem consists in making HP stand out from its good companions, without being able to appeal either to the former's acceptability (since all the other principles are equally acceptable) or to its mathematical strength (since each of the other principles can be used in place of HP in the derivation of Frege's Theorem).

10 See *Mancosu* 2016, p. 172.
11 See *Mancosu* 2016, p. 184.
12 Recall in particular that 'Fin(F)' can be defined in second-order logic; cf. fn. 6 above.
13 *Mancosu* 2016, p. 178.
14 *Mancosu* 2016, pp. 197–9.

Now, the conjunction of (1) and (2) above entails that BP and its like are *on a par* with HP. On the other hand, Mancosu also suggests that, since NUM does not satisfy (1) and (2), it might ultimately be dismissed by the neologicist (*Mancosu 2016*, Sect. 4.6). Moreover, Mancosu argues that no (good) principle can preserve PWP in its *totality*,[15] since no abstraction principle similar to HP which *fully* preserves PWP is satisfiable: the relation on the right-hand side of any such principle would indeed partition the domain of concepts into more equivalence classes than there are objects in the domain.[16]

It is however worth noticing that the neologicist rebuttal of NUM would beg the question against the supporter of numerosities. Indeed, the neologicist might claim that since the right-hand side of NUM cannot be formulated by means of second-order logic alone, then NUM is not a viable alternative to HP. The advocate of **TN** might reply, however, that since NUM is a consistent alternative to HP, the ordinary notion of cardinal number cannot be characterised by a principle whose right-hand side is purely logical, contrary to what the neologicists claim. As a consequence, it fully makes sense to also view NUM as a principle of numerical abstraction potentially able to challenge and undermine the neo-logicist's claim that HP is inherent in our ordinary arithmetical reasoning; our overall 'Cantorian' strategy, therefore, will consist both in defusing Mancosu's version of the 'Good Company problem', but also in challenging NUM's suitability as an alternative principle of numerical abstraction.

3. Hale's Defence of Hume's Principle

In his review of Mancosu's book, Bob Hale sketches a response to the Good Company problem; he writes:

> Some have thought that if one can coherently take different countably infinite sets to differ in size, it follows that Hume's Principle cannot be a conceptual truth. But that inference seems far too swift. It assumes, not only that the theory of numerosities captures *a* concept of cardinal number, but that it seeks to capture a *single* concept of cardinality of which the Cantorian conception, enshrined in Hume's Principle, offers a rival account. Both assumptions are clearly resistable. An obvious response, open to the neo-Fregeans, is to reject the second, even if they accept the first. Mancosu is more cautious – the theory of numerosities and the Cantorian theory are 'not in conflict. Conflict emerges only if both notions are taken to explicate the same intuitive notion of size', and they need not be so taken. He does, however, view the former as offering an alternative conception of cardinality to Cantor's, and so rejects Gödel's claim that the Cantorian conception is inevitable – Gödel's argument, he contends, begs the question by assuming that how the elements of a collection are grouped must not affect the result of counting them. While much more needs to be said than I can say here, it seems to me that Gödel was right on this point. Indeed, getting different numerosities for different countable infinite collections depends not only upon grouping their members in certain ways, but also on taking the groups in a certain order – but it is of the essence of cardinal as opposed to ordinal numbers that the cardinal size of a collection does not depend upon how its members are ordered.[17]

[15] BP preserves at least some of the infinite instances of PWP – for example, BP entails, as we have seen, that the cardinality of $\mathbb{N}$ *is not equal* to the cardinality of $\mathbb{N} - \{1\}$.

[16] See *Mancosu 2016*, Appendix 4.10, pp. 200–1; *Mancosu and Siskind 2019* show, moreover, that any principle which fully satisfies PWP is formally refutable in second-order logic.

[17] *Hale 2018*, p. 162; the references to Mancosu's book have been suppressed.

Hale claims that the neologicist can respond to the Good Company problem in two ways: by (a) denying that the the ordinary notion of cardinal number corresponds to a *unique* abstraction principle, or by (b) denying that there is *any* concept of cardinal number that corresponds to NUM and the other principles. Let us examine each of the two strategies in more detail.

(a) goes in the direction of a form of *conceptual pluralism*:[18] the neologicist could argue, for example, that the informal notion of cardinal number admits of different *precisifications*, each of which is captured by a different principle. For example, HP might be one such precisification, and BP another one. However, this first strategy is very risky, as it could potentially force the neologicist to claim that, like the naïve concept of set, the ordinary notion of cardinal number is ultimately inconsistent, insofar as it would conflate equinumerosity, corresponding to HP, and the Part-Whole Principle, corresponding to NUM.[19]

(b) might, *prima facie*, seem even more radical, as it consists in denying that NUM and the other good companions introduce a *genuine* concept of cardinal number. However, (b) has two main advantages. First, it eliminates the Good Company problem altogether. To see this, suppose that HP and NUM did in fact introduce different concepts; then no conflict would arise, since the two principles could be seen as introducing different kinds of objects. Second, the neologicists can deploy (b) without being thereby committed to (a). In particular, (b) is compatible with Gödel's claim that the Cantorian conception is 'inevitable', and so that the concept of cardinal number is captured uniquely by HP.

To sum up: Hale's response (b) to the Heck-Mancosu argument is that NUM – like all the other good companions – fails to introduce a *genuine* concept of *cardinal number*. If Hale's defence is sound, then the neologicist can deny not only that HP and NUM define rival notions of cardinality, but also that there is any notion of cardinality that is captured by NUM. In what follows we will show that Hale's defence of HP can be implemented in a uniform way, by using a Cantorian conception of abstraction. We will then argue that Cantor's conception of abstraction can be used to produce an argument for HP's uniqueness.

4. Cantor's Abstractionism

4.1. Two Accounts of Number

We now proceed to present Cantor's *abstractionist* definitions of number.[20]

As a preliminary to that, we think it is useful to quickly review Cantor's construction of the transfinite number-class hierarchy based on the *ordinals*.

Cantor first 'discovered' the transfinite ordinals as a consequence of defining the iteration of *point set derivation* in work addressing the *unique* representations of functions through *trigonometric series*.[21] As an extrapolation and generalisation of the latter process, he was led to produce the earliest instance of the sequence of ordinals *past* the natural

[18] See *Sereni et al. 2021*, §3.1 for a discussion of this option.

[19] *Heck 2011*, p. 264 argues, however, that 'there looks to be no basis for the view that ordinary arithmetical thought harbors inconsistency'.

[20] With an important *caveat*: as stressed by *Hallett 1984*, p. 123, 'Cantor's discussions of number are (to say the least) quite vague; one is therefore forced to *interpret* him.'

[21] For details on the progressive unfolding of Cantorian set theory from its origins in analysis, see *Hallett 1984*, *Lavine 1994*, Ch. 1, and *Kanamori 1996*, pp. 2–9.

numbers:

$$\omega, \omega + 1, \ldots, \omega + \omega, \omega + \omega + 1, \ldots, \omega^{\omega}, \ldots$$

which he gradually construed as the collection of all *order-types* of sets of natural numbers.[22] Subsequently, he realised that, by gathering all such ordinals together, he could produce a new collection of ordinals, provably greater, in cardinality, than the collection of all *finite* ordinals. He had produced a new *number-class*, beyond that of the natural numbers, whose *power* he indicated with (II) (whereas (I), as is clear, is the power of the number-class of all *finite* ordinals). Now, *qua* well-ordered set, (II) has an order-type, which, again, is an ordinal, ω_1.[23] Through the combination of what he came to view as 'generating principles', Cantor was, thus, able to envision the 'completed' sequence of transfinite ordinals (Ω) as follows:

$$\Omega : 0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega_1, \omega_1 + 1, \ldots, \omega_2, \ldots, \omega_{\omega}, \ldots$$

where each *transfinite* number-class may be viewed as the collection of all ordinals which have a strictly smaller power. So, (II) is the collection of all ordinals whose power is (I), (III) that of all ordinals whose power is (II) and so on. This construction produces, among other things, a very straightforward correspondence between ordinals and cardinals.[24] The symbols (I, II, III,...) were subsequently replaced by the $\aleph$-notation: $\aleph_0, \aleph_1, \aleph_2 \cdots$.

Now, as Michael Hallett explains, one might just be content with seeing Cantor's *cardinals* as the *powers* of the number-classes they 'enumerate', thus ultimately taking Cantor's concept of number to be nailed to the *ordinal sequence*.[25]

However, in previous works (*Cantor 1878*, *Cantor 1879* and, especially, *Cantor 1885*), Cantor had given a more general, and, one should add, logically more perspicuous, definition of number, which does not rely upon the cumulative number-class hierarchy. Moreover, later, in the 1887-88 *Mitteilungen* (collected in *Cantor 1890*) and in *Cantor 1895*, Cantor more explicitly enunciated his two *abstractionist* definitions of number.

What we shall call Cantor's *first abstractionist definition of number* may be summarised, using more or less Cantor's own words in *Cantor 1885*, as follows:[26]

A cardinal number is the *general concept* under which fall all and only those sets which are *equipollent* to a given set.

C_1 above may be formally stated as the *equinumerosity principle* which is known as Cantor's Principle (CP), and which may be seen as an alternative, although equivalent, formulation of HP:[27]

$$card(x) = card(y) \leftrightarrow x \cong y, \qquad \text{(CP)}$$

where the '$\cong$' sign precisely indicates the 'equinumerosity relation', that is, that two sets x and y have the same number of elements (which, in turn, is equivalent to the existence of a function $f : x \to y$ which is one-to-one and onto between them).

[22] It is worth recalling that it was only with *von Neumann 1923* that the interpretation of the ordinals as *sets of their predecessors* (with 0 being the null set) came to the fore.

[23] The well-ordering principle was formulated by Cantor about the same time as he produced the sequence of power-classes (cf. *Cantor 1883*, in *Ewald 1996*, p. 886, where Cantor declares it a 'law of thought' [*Denkgesetz*]).

[24] Although the straightforward correspondence breaks at the level of the ωth number-class (which, instead, needs to be defined as the *union* of all ordinals which belong to all $n < \omega$ number-classes).

[25] *Hallett 1984*, pp. 140–1. For more on Hallett's ordinal-reductionist interpretation, see §4.2.1.

[26] Cf. *Cantor 1885*, in *Ebert and Rossberg 2009*, p. 346.

[27] Note, though, that, to obtain the formal equivalence between CP and HP, one has to construe Frege's concepts (for which see fn. 3) as *sets*.

What we may call Cantor's *second abstractionist definition of number* originates from *Cantor 1890*, but, as we have already said, was more determinately fixed in *Cantor 1895*, the last major contribution made by Cantor to set theory, and the most systematic exposition of his studies. The passages from which it can be extracted are worth re-producing in full:

> We will call by the name "power" or "cardinal number" of M the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given. We denote the result of this double act of abstraction, the cardinal number or power of M, by $\overline{\overline{M}}$. Since every single element m, if we abstract from its nature, becomes a "unit", the cardinal number $\overline{\overline{M}}$ is a definite aggregate composed of units, and this number has existence in our mind as an intellectual image or projection of the given aggregate M.[28]
> Every ordered aggregate M has a definite "ordinal type" or more shortly a definite "type" which we will denote by $\overline{M}$. By this we understand the general concept which results from M if we only abstract from the nature of the elements m, and retain the order of precedence among them. Thus the ordinal type $\overline{M}$ is itself an ordered aggregate whose elements are units which have the same order of precedence amongst one another as the corresponding elements of M, from which they are derived by abstraction.[29]

We may condense Cantor's assertions as follows:

C_2. (1) Given a set M, its *ordinal* set, $\overline{M}$, is a special set, which originates from an act of abstraction from the *nature* of the elements of M. (2) Given a set M, its *cardinal* set, $\overline{\overline{M}}$, is a special set which originates from a *double* act of abstraction, from the *nature* and *order* of the elements of a set M. In both (1) and (2), the elements $m \in M$ are turned into *units* [*Einheiten*], members, respectively, of the *ordinal* and of the *cardinal set* of M.

C_2 has historically been seen as especially problematic, and has, consequently, received far more attention than C_1. By explicitly distancing itself from the generally held views about Cantor's abstractionist strategy (see fn. 30), our interpretation of Cantor's abstractionism in the next section aims to argue that C_1 and C_2, albeit independent, are, in fact, mutually connected. In particular, we will argue that C_2 represents an 'explanation' of C_1, and, most crucially, an explanation of what *numerical abstraction* should be like.

4.2. The Explanation Thesis

Our 'explanation thesis' is based on a more literal construal, than has been done so far, of what Cantor says in both *Cantor 1890* and *Cantor 1895*. Many scholars reject a face value interpretation of Cantor's statements, on the grounds that such statements, as pointed out by Frege himself, cannot be made sense logically.[30] In particular, the most controversial

[28] *Cantor 1895*, p. 86.
[29] *Cantor 1895*, p. 111–2.
[30] Frege's hostility towards Cantor's theory of *units* is transparent in his review of *Cantor 1890*, for which see *Frege 1892*, but is already conspicuous in the *Grundlagen* (*Frege 1974*); in both works Frege famously points out that if any two units u_1 and u_2 are mutually discernible, then they must at least retain *some* of their defining properties, so abstraction hasn't fully gone through, after all; if they are not, then they are just reducible to a *single* unit, which is clearly not what Cantor wants. For further considerations on Frege's objections, see *Fine 1998*, pp. 602ff. and *Tait 1997*, especially §8 (reproduced in *Tait (2005)*, pp. 232ff.). Fine avows a full-blown 'ontological' interpretation of Cantor's abstractionism, whereas Tait declares C_2 'ill-conceived', but coherent.

point of Cantor's account is that the latter would imply the creation of *new* objects, whose properties would be mutually contradictory, and whose nature would be unintelligible.

Now, a potential alternative construal of Cantor's theory rather views 'units' as a purely descriptive, ontologically non-committal, medium to convey the explanation of what numerical abstraction consists in, a construal which, inevitably, rules out the view that units would be 'new entities'.

To be sure, the point is, to say the least, controversial: there are passages in Cantor's works, where it would look like that the ordinal and cardinal sets have a more substantial, and definite, character. For instance, in *Cantor 1895* Cantor construes CP as a two-way theorem: that is, he first proves that, if two sets are equinumerous, *then* they have the same cardinal set, and, second, that, if two cardinal sets are equal, then they are equinumerous. For this second part of the proof, he needs to use the fact that $M \cong \overline{M}$, thus, he has to treat $\overline{M}$ as a definite set.[31]

However, there are other passages, where he just construes the equivalence of one cardinal to another as just involving the idea of the *substitutional invariance* of the elements of a set within the process carried out by the pairing function. In the aforementioned passage of *Cantor 1895*, he explains:

> [A]ccording to the above definition of power [that is, C_2, *our note*], the cardinal number $\overline{\overline{M}}$ remains unaltered if in the place of each of one or many or even all elements m of M other things are substituted.[32]

Earlier, he had stated that:

> The cardinal number M of a set M remains unchanged according [to the abstraction idea], if other things are substituted in place of the elements $m, m', m'', \ldots$ of M. If now $M \sim M_1$, then there is a correspondence by which the elements m, m', m'', $\cdots$ of M correspond to the elements m_1, m_1', m_1'', $\cdots$ of M_1. One can think of the elements $m_1, m_1', m_1'', \ldots$ of M_1 as substituted for the elements m, m', m'', $\cdots$ all at once. Thereby the set M is transformed into the set M_1, and with this transformation nothing in the cardinal number is altered, then $\overline{\overline{M}} = \overline{\overline{M_1}}$.[33]

Now, we think that it is precisely to convey, and explicate, this idea of 'transformation' inherent in his *first* abstractionist definition of number that 'units' have been used by Cantor in his *second* definition; on this point we wish to dwell on in more detail.

The key feature of Cantor's approach to defining numbers, both in C_1 and C_2, is the idea of the one-to-one correspondence between the elements of sets, which is, in turn, based on the use of a *pairing function*. Now, functions usually associate objects to other objects in a law-like manner, but Cantor's pairing functions are no determinate laws: *any* bijective function is as good as any other for the abstraction process.[34] This means that, given two sets M and N, and a pairing function f, f will pick any $m \in M$ and pair it with any $n \in N$ in a completely unpredictable way. So, the first and foremost feature of pairing functions is that they are 'free-form transformations' of objects of one set into objects of another set. One could further express this property of Cantor's abstraction by saying that a fundamental aspect of the pairing process is that, for it to go through, the *nature* of

[31] Cf. *Cantor 1895*, p. 88.

[32] *Cantor 1895*, p. 88.

[33] *Cantor 1890*, p. 413.

[34] For the sake of completeness, it should also be noted that abstraction, in itself, as pointed out by Russell, does not guarantee the uniqueness of the cardinality function, since there are in principle infinitely many functions that satisfy CP (and HP); therefore, one only gets 'Card(x) with respect to function f' for any suitable function f. We thank an anonymous referee for raising this point.

the elements of the sets involved is entirely irrelevant. One could maybe express this in *category-theoretic* terms, by noting that what matters most to Cantor is the *isomorphism* between sets of objects, not the objects themselves.[35]

Now, the theory of units comes into play and is deployed by Cantor precisely in such 'functional/transformational/category-theoretic' context, whereby certain (pairing) functions produce *isomorphic* transformations of sets into other sets in the most abstract and general possible way. Let us see how.

Let us consider the *cardinal set* of a set M. Such a set, $\overline{\overline{M}}$, as we know, will consist, after the abstraction process has been carried out, of units, but the abstraction process, as we have seen, has crucially availed itself of a pairing function, so the reduction of the elements of M to units may just be construed as follows: whenever the set M is paired with any other set *equinumerous* to it, then it may practically come to be 'seen' as a set composed of *units*, given that the pairing process carried out by the pairing function will ignore any other property of M except for *numerical distinctness*. This is precisely what the term 'unit' is, ultimately, supposed to evoke: *pure numerical distinctness*, as well as *substitutional invariance* of the elements of a set in the context, and for the purposes, of the pairing process. On these assumptions, then, we believe that C_2 provides an explanation of the abstraction process inherent in C_1, in particular, of the procedure involved in it.

However, as anticipated, by further elaborating on this interpretation, we take Cantor's conception to stand out as a more general explanation of what is entailed by *numerical abstraction*. In particular, according to Cantor, numerical abstraction incorporates the following two features:

(1) Each element of a set M is 'transformed' by the abstraction process into the element of another set N with which it is paired (each $m \in M$ is 'transformed' into an $n \in N = f(m)$)
(2) The nature of the elements of M (and N) is irrelevant, as any element of M may be substituted by any element of any other (equinumerous) set (*substitutional invariance*)

We now proceed to examine Gödel's reconstruction of Cantor's conception, which further zeroes in on the aforementioned properties of numerical abstraction.

5. Gödel's Minimal Account of Abstraction

Gödel's Cantor paper (*Gödel 1947*, then revised *Gödel 1964*) deals with fundamental philosophical questions related to Cantor's Continuum Problem, in particular, with its meaningfulness as a mathematical problem.[36] As is known, the point of view articulated in the paper (i) advocates, on the one hand, the definiteness of the Problem as a consequence of the definiteness of Cantor's *concept of cardinality* and, on the other (ii) holds that the problem would retain its meaning, even if, as happened afterwards, it were shown to be unsolvable in **ZFC**.

We will just limit ourselves to taking into account (i). Crucially for our purposes, Gödel's line of argumentation is meant to show that not only does Cantor's definition of '(transfinite) cardinal' make full sense, but also that it is *inevitable*, so long as one accepts Cantor's *abstractionist* strategy. Among other things, Gödel explains that:

[35] For further details on cardinality assignments arising in connection with the idea of *isomorphism*, see the considerations in *Lawvere and Schanuel 2009*, pp. 106–7.

[36] The Continuum Problem is the problem of what is the (transfinite) cardinality of the continuum (of $\mathbb{R}$). Cantor's Continuum Hypothesis was that $card(\mathbb{R}) = \aleph_1$. For an overview of the status of the problem, see, for instance, *Koellner 2019*.

> For whatever 'number' as applied to infinite sets may mean, we certainly want it to have the property that the number of objects belonging to some class does not change if, leaving the objects the same, one changes in any way whatsoever their properties or mutual relations (e.g. their colors or their distribution in space). From this, however, it follows at once that two sets (at least two sets of changeable objects of the space-time world) will have the same cardinal number if their elements can be brought into a one-to-one correspondence, which is Cantor's definition of equality between numbers. For if there exists such a correspondence for two sets *A* and *B* it is possible (at least theoretically) to change the properties and relations of each element of *A* into those of the corresponding element of *B*, whereby *A* is transformed into a set completely indistinguishable from *B*, hence of the same cardinal number.[37]

In the quote above, Gödel highlights two aspects of Cantor's abstractionist definition of number which we have scrutinised in the previous section: the first one is what we have called *substitutional invariance*, that is, the fact that for a definition of cardinal number to be available, one has to conceive of the nature of the elements of a set as being irrelevant; the second aspect is that such an *invariance* lies at the core of the definitional process qua based on the use of *functions*. The novelty of Gödel's approach consists in the fact that it replaces the descriptive medium of 'units' with its 'functional' equivalent, namely the idea of *invariance* under the pairing process, and that it also sharply emphasises the equivalence between one-to-one correspondences and invariance.

Thus, Gödel's account outlines *necessary and sufficient* conditions for *numerical abstraction*, which, taken jointly, constitute what we shall call the Minimal Account of Abstraction:

Gödel's Minimal Account of Abstraction (GMAA). Numerical abstraction entails the satisfaction of two criteria:

(1) The use of a *pairing function*
(2) The indifference to the *nature* of the objects involved

Some clarifications are in order. The qualification of 'minimal' for this account should be interpreted as follows: no *smaller* set of conditions exists, which would produce as *good* a form of numerical abstraction. However, such an account is 'minimal' also as opposed to other accounts of abstraction, which we believe to be 'non-minimal', and which we shall review later on.[38]

Now, although generally accepted as cogent, GMAA may, in fact, be problematic.[39] To begin with, does it really show that Cantor's cardinals are *inevitable*? In particular, Mancosu has pointed out that GMAA has, in fact, been purposefully crafted by Gödel in such a way as to straightforwardly obtain that Cantor's notion of cardinality is the only correct one, so the introduction of GMAA really begs the question of why Cantor's cardinals are inevitable.[40]

[37] *Gödel 1947*, p. 254.
[38] In his introduction to *Gödel 1947* in *Gödel 1990*, p. 160, Gregory Moore has also pointed out the 'minimality of requirements' involved in Cantor's definition of cardinal number.
[39] For Mancosu's position, see the the next few lines. *Parker 2013* finds Gödel's arguments somewhat lacking. He stresses that PWP may be as intuitive as CP, and that the reasons why the theory of numerosities is not a serious contender of set theory is that it violates principles which are far more cogent than CP such as the invariance of *rigid* transformations in geometry.
[40] *Mancosu 2016*, p. 147.

A full case for GMAA is beside the scope of this paper, but the considerations which follow at least show that Mancosu's objections can be adequately dealt with.

In our view, there are independent reasons to support GMAA. To begin with, GMAA is fully general in its applicability, insofar as it fosters a *uniform* treatment of finite and infinite cardinals.[41] One may object that, in order to attain this, one has to effect a 'reduction' of numbers to sets, and then thrive on the *bi-interpretability* of the former with the latter, a point which is, to say the least, contentious. However, as controversial as set-theoretic reductionism might be, it has never been questioned in the debate between neologicists and their critics about the status of HP. So, as far as the purposes of such a debate are concerned, numbers may *really* be sets.[42]

Another reason why one should view GMAA as independently motivated is of a more general epistemological character. The reader might have already noticed that while, on the one hand, GMAA pins down 'minimal' criteria to produce a good form of abstraction, it also implies that the extent of such an abstraction be, in fact, 'maximal'. This is because, as we have seen, GMAA postulates the reduction not only of all spatio-temporal objects and relations, but also of all mathematical objects and relations to just one simple type of relation, that of *equinumerosity*. In a word, GMAA is *simple, maximal* and *universally applicable*, and all of these stand out as epistemologically attractive features of a mathematical theory.[43]

So, here, finally, goes our Cantorian argument: there are persuasive reasons to believe that GMAA represents the correct conception of numerical abstraction, and this fact bolsters neologicists' view that HP has a stronger claim to being seen as privileged among all principles of numerical abstraction.

In conclusion of this section, we somewhat anticipate our full response to the Good Company problem, by showing how 'Mancosu's principles' badly fail to satisfy GMAA. In brief, all of these treat *finite* concepts and *infinite* concepts *differently*. Peano's Principle assigns the same cardinal number to two concepts if they are both infinite, or equinumerous; so, in order to apply PP one would need to determine in advance whether a concept is finite or infinite (in the former case, GMAA would hold, in the latter, it would not). Boolos' Principle, on the other hand, as we have seen in Section 2, assigns the same cardinal number to all infinite and co-infinite concepts, and a different cardinal number to all the concepts that are infinite but co-finite, whereas each instance of BP-n assigns the same cardinal number to each infinite concept with co-finality n, and a different cardinal number to all infinite concepts with co-finality $\neq n$. Overall, then, both PP and BP violate GMAA, and this fact provides us with concrete means to distinguish HP from its (purely logical) companions, hence with a reply to Mancosu's version of the Good Company argument.

6. Numerosities and GMAA: A Response to the 'Good Company' Problem

We shall now proceed to use GMAA to reject the view that NUM, the principle of numerical abstraction formulated in the context of **TN**, stands on an equal footing with HP.

Our plan is as follows. We will first provide the readers with further details about *numerosities*.[44] Afterwards, we shall formulate NUM and discuss its peculiarities. Finally, we shall deploy our Cantorian argument as a response to the Good Company problem as based on NUM.

[41] The Cantorian equivalent of this is what Hallett, in *Hallett 1984*, pp. 32–40, calls 'Cantor's *finitism*'.

[42] Cf. e.g. *Hale and Wright 2001*, p. 392.

[43] *Mancosu 2016*, p. 190 attributes this suggestion to an anonymous referee; see also *Hale 2018*, p. 165.

[44] Our exposition of the basic features of **TN** is mostly based on *Benci and Di Nasso 2003* and *Benci et al. 2006*; for further developments of the theory, see, among other works, *Benci and Freguglia 2019*, *Benci et al. 2020*.

As we already know from Section 1, numerosities capture intuitions about numbers and sets based on the Part-Whole Principle:

$$x \subset y \rightarrow card(x) < card(y) \tag{PWP}$$

Now, PWP is, in fact, a descendant of the century-old axiom known as Euclid's Axiom:

Euclid's Axiom. *The whole is greater than the parts*.[45]

A straightforward consequence of PWP is that, for instance, all *proper subsets* of $\mathbb{N}$ have a smaller cardinality than $\mathbb{N}$ itself. Thus, PWP does justice to the intuition (briefly discussed in Section 1) that the *density* of $\mathbb{N}$ is greater than that of its proper parts. We shall see in a moment that NUM also obeys this principle (and intuition about *densities*).

TN fosters a different way of counting. By CP (and HP), counting means to draw a correspondence between *equinumerous* collections of objects (concepts). **TN**, on the other hand, views counting as the process of adding up *groups* of objects progressively. In order to carry out this process, **TN** needs to specify 'canonical' ways of counting the elements of a set through suitable *groupings*.[46] The grouping of the elements of a set is, in turn, formally accomplished by a labelling function:

Definition 1 A labelling is a pair $< A, \ell(A) >$, where A is the domain of objects and $\ell(A)$ is a function from A to $\mathbb{N}$, which assigns a number (the 'label') to each object in A.

Given a set A, and using labelling, we can thus produce a *partition* of A, so as to obtain the *partial sum*: $A = A_0 + A_1 + A_2 + \cdots$ (where the elements of each A_n are specified by $\ell(A)$). Now, labellings may vary, so *not any* labelling will do and, in fact, **TN** uses just one labelling, which is called 'canonical', for all sets of ordinals α: $\ell(\alpha) = \alpha$ (in particular, $\ell(n) = n$, with $n \in \mathbb{N}$).[47] **TN**'s notion of *cardinality* of a set A is, thus, defined as follows:

$$card(A) = card(A_0) + card(A_1) + card(A_2) + \cdots$$

Using the canonical labelling, the equation above yields a *series*, equivalently, an *increasing sequence of naturals*, whose each initial segment may be seen as an 'approximation' to $card(A)$. Thus, the cardinality of a set, for some labelling, is equivalent to the value of its 'approximating function':

$$\#A = \#A_0 + \#A_1 + \#A_2 \cdots + \#A_n + \#A_{n+1} \cdots$$

where $\#A_n$ is $card(A_n)$. More rigorously, we define the approximating function as follows:

Definition 2 (Approximating Function) An approximating function is a function $\gamma_A : \mathbb{N} \rightarrow \#A_n$.

Now, two approximating functions are equal if their corresponding terms, taken one by one, are equal. There is, however, a complication, insofar as there may be approximating

[45] In Euclid's *Elements*, I, the axiom features, in fact, as Common Notion 5.

[46] For instance, let A be a (denumerably) infinite collection of boxes, each of which containing one object. According to both CP and NUM, the number of objects of A will be ω. But if we add 10 objects to the first box, and use CP to count, the number of objects will still be ω, whereas, if we use NUM, the number of boxes will be: $10 + \omega \neq \omega$.

[47] Extensions of numerosities to sets whose cardinality is $\geq \aleph_0$ has only recently been carried out. For further details, see *Benci et al.* 2020.

functions whose terms are *all equal* only from a certain point *onwards*. In that case, intuitively, any two such functions should still be taken to be equal, but we have to proceed in a more rigorous way in order to take into account all possible cases.

As for the construction of $\mathbb{R}^*$ in non-standard analysis, for the construction of the set of all numerosities one picks up an *ultrafilter* (a 'selective' ultrafilter) $\mathcal{U}$ on $\mathbb{N}$, and get $num(A) = num(B)$ if and only if the set of indices I for which the terms of the approximating sequence of A are equal to the terms of the approximating sequence of B is in $\mathcal{U}$.[48] Therefore, numerosities may more correctly be thought of as *equivalence classes* of increasing sequences of natural numbers.[49] So, we may, finally, rigorously formulate the theory's two fundamental notions:

Definition 3 (Numerosity) Given a labelled set A, $num(A) = [\gamma_A]_{\mathcal{U}}$, where $[\gamma_A]_{\mathcal{U}}$ is the equivalence class of A induced by $\mathcal{U}$.

Definition 4 (Set of Numerosities) The set $\mathcal{N}$ of numerosities is defined as follows: $\mathcal{N} = \{[\phi]_{\mathcal{U}} \mid \phi : \mathbb{N} \to \mathbb{N}\}$, where ϕ is an increasing sequence of natural numbers.

We are now ready to state NUM:

$$num(A) = num(B) \leftrightarrow A \sim_{\mathcal{U}} B, \qquad \text{(NUM)}$$

where $\sim_{\mathcal{U}}$ is an equivalence relation defined as follows:

$$A \sim_{\mathcal{U}} B =_{df} I = \{n \in \mathbb{N} : \#A_n = \#B_n\} \in \mathcal{U}$$

and where, as we know, $\#A_n$ and $\#B_n$ are the nth terms, respectively, of $\#A$ and $\#B$, the approximating functions of A and B.

Let us examine NUM a bit more closely. As in CP, the left-hand part of the bi-conditional expresses the identity of two cardinal numbers. The right-hand part: '$A \sim_{\mathcal{U}} B$', on the other hand, expresses a peculiar *equinumerosity* relationship, so what we really need to check is that the idea of *equinumerosity* expressed by NUM is sanctioned by GMAA. As the readers might have already guessed, this cannot be the case.

As we already know, GMAA states two fundamental criteria to be met by numerical abstraction: (1) one should just use one-to-one correspondences for that, and (2) the properties of the objects involved should not count.

Now, (1) ceases to be valid in **TN**, as the notion of numerosity also involves comparing the terms of specific *sequences of numbers*. More formally, ascertaining that two sets A and B have the *same* numerosity consists in verifying that the set of indices of the terms of $\#A$ and $\#B$ which are equal is in $\mathcal{U}$. But this, crucially for the violation of (2), entails that properties of sets *do count* and, in fact, the whole point of using numerosities consists in 'preserving' such properties for counting purposes. For instance, by NUM's lights, $\mathbb{N}_{odd}$, the set of all odd numbers must be smaller, in size, than $\mathbb{N}$. But for this to be true, one needs to take into account 'oddness' as a defining property of the objects in $\mathbb{N}_{odd}$. *Substitutional invariance*, the idea lying at the heart of GMAA is, thus, flagrantly violated by NUM: if one replaces all the elements of $\mathbb{N}_{odd}$ with the elements of another set, say, in particular, with the elements of $\mathbb{N}$, one does not obtain the same numerosity as the one of $\mathbb{N}_{odd}$.

[48] Cf. *Goldblatt 1998*, pp. 15–17. The ultrafilter construction for **TN** is described in *Benci and Di Nasso 2003*, pp. 60ff.

[49] Surprisingly enough, the existence of the (selective) ultrafilter needed to produce a numerosity function is independent from **ZFC**, while it is known that it is directly implied by **ZFC** plus the combinatorial principle known as Martin's Axiom (see *Jech 2003*, p. 272, and *Benci and Di Nasso 2003*, p. 9).

In a sense, this was entirely predictable, as any abstraction principle which obeys PWP must be able to discriminate among different ‘proper parts’ of a set, but the heavy price at which this comes was not as predictable: in order for PWP to be meaningfully implemented in a different way of counting, that expressed by NUM, one needs to *add*, not *abstract from*, features of sets. In particular, as we have seen, **TN** employs labellings, without which the entire approximating function-based counting process is ineffective. So, what goes wrong with NUM, by GMAA’s lights, is that the principle involves prominently *anti-abstractionist* features: in a sense, in the face of (logical) appearances, NUM is not even an abstraction principle.

NUM may also be seen as running counter to the preference for the *theoretical minimality* encapsulated by GMAA and, thus, as having, in this respect, also *anti-foundational* features. Among other things, **TN** is a very complicated theory, which uses resources which lie beyond the standard axioms of set theory (see fn. 49): while, on the one hand, this is not an issue *per se*, one the other, if one identifies ‘ordinary maths’ with (fragments of) **ZFC**, then one should not take numerosities to belong to ordinary maths.

To sum up, here is the main outcome of this section: a new abstraction principle, NUM, motivating the adoption of a theory of cardinalities alternative to Cantor’s theory, **TN**, cannot be seen as being on an equal footing with CP, insofar as NUM manifestly violates GMAA, which we have taken to be the right conception of numerical abstraction.

The rejection of NUM as a suitable abstraction principle, along with the rejection of ‘Mancosu’s principles’ mentioned in Section 1, have thus, finally provided us with a robust and, in our view, penetrating response to all instances of the Good Company problem introduced in Section 1.

7. Conclusions

We have formulated a Cantorian argument for HP. We have first distinguished between two conceptions of numerical abstraction formulated by Cantor: C_1 – according to which the cardinal number of a set is the general concept under which fall all and only those sets which are equinumerous to that set – and C_2 – according to which cardinal numbers are attained by abstracting away from all properties of the elements of a given set but their numerical distinctness, and we have suggested that C_2 provides an explanation of what *cardinal abstraction* is, and, in particular, of what C_1 consists in. We have then further illustrated the features of Cantor’s abstractionism through using Gödel’s Minimal Account of Abstraction (GMAA), and shown that non-Cantorian assignments of cardinality, and especially the one given by **TN**, do not satisfy GMAA’s constraints on good abstraction.

As previously made clear, our Cantorian argument for HP can be seen as a way of implementing Hale’s rebuttal of the Heck-Mancosu ‘Good Company’ objection. The objection is that HP is not an *implicit definition* of our concept of (cardinal) number because alternative assignments of cardinalities to infinite concepts are (at least conceptually) possible. As we have seen, Hale’s defence of HP consists in arguing that the principle NUM, arising in **TN**, does not provide a concept of cardinal number since, in violation of GMAA, ‘getting different numerosities for different countable infinite collections depends on taking the groups in a certain order’.

We conclude this summary with the following consideration. While, on the one hand, our argument may substantially help neologicists to dispel the threat represented by the Good Company problem, on the other hand, it could also bring to the fore a potential dilemma for them. Indeed, if neologicists accept the argument as it stands, then they must also accept the Cantorian conception of *numerical abstraction* that underlies it. However, Frege himself vocally expressed his distaste for such conception (see fn. 30). So, in order to adopt the argument and solve the Good Company problem as they wish (that is, by defending HP), the neologicists must dismiss Frege's own misgivings about Cantor's conception of abstraction. If, by contrast, the neologicists follow Frege in his rejection of Cantorian abstraction, they cannot appeal to GMAA in order to defend HP, and so they will have to face up with the Good Company problem again.

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