From Bolzano to Frege: A Cantorian Path

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The Problem

It is well known that Frege held a peculiar view about *concepts*, and that these played a fundamental role in his doctrines. However, little attention has thus far been paid to the roots of Frege's conception, while it's almost universally assumed that Frege did not draw upon other conceptions or authors.

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The goal of the paper is to explore Frege's conception of *concepts* through the interpretative framework of the broad and robust segment of *anti-psychologistic*, *objectivist* and *pre-logicist* XIX century philosophical tradition, which was fundamentally initiated by Bernhard Bolzano.

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However, our main focus in the paper will not be Bolzano, but rather one of the main followers, so to speak, of Bolzano's conceptions, that is, Georg Cantor. More specifically, we shall argue that some very characteristic Fregean conceptions draw upon Cantorian conceptions which, in turn, are due to or were inspired by Bolzano's conceptions.

Goals of the Paper

- We will offer a fresh interpretation of *Frege*'s theory of concepts through the lens of *Cantor*'s conception;
- We will point to the existence of an *indirect* path *from Bolzano to Frege*, whose main stopping point is represented by Cantor's conceptions
- We will point to *structural* similarities among the three authors
- We will aim to provide an account of the robustly *Cantorian features* of Frege's doctrines, in particular in the following areas:

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    1. *Objectivity* of concepts (of mathematical entities)
    2. The notion of 'set' (and set-theoretic reductionism)
    3. *Abstractionist* strategies for definitions
    4. Notion of *conceptual dependence*

Bolzano's *Wissenschaftslehre*

Bolzano's *Wissenschaftslehre* (henceforth, *WL*) contains an account of concepts which has exerted an enduring influence on generations of later thinkers. Further details on concepts may be found in the Introduction to the unfinished *Grössenlehre* (*Of The Mathematical Method*).

Among the views championed in both works, one should mention:

1. The idea that *propositions*, *representations*, *concepts* have a unique status, as they are not reducible to: (i) linguistic expressions; (ii) mental constructs
2. Propositions-in-themselves are made of ideas-in-themselves (*Vorstellungen an sich*)
3. Ideas-in-themselves may either be *intuitions* or *concepts*, simple and singular ideas-in-themselves
4. Both are the building blocks of science (language). Concepts are simple and singular ideas which are not made of intuitions

Bolzano on Concepts/Cont'd

What is particularly interesting here is the fact that concepts should not be reduced to their psychological correlate (anti-psychologism), and that it gives rise to a notion of purely conceptual dependency and derivability.

Definition

A conceptual proposition is a proposition which contains only *concepts*.

Example: 'The order of the terms of the product $a \times b$ does not change its result.'

Bolzano then proceeds to introduce the notion of *consistency*, *validity* and *derivability*. It is useful, for our purposes, to recall the notion of *derivability*.

Conceptual Dependency

Definition

xxx

Cantor on Concepts

Fundamental features of his account (for which see, fundamentally, [?]) are:

1. Concepts are to be construed *platonistically*
2. Concepts are connected to the *immanent* (as opposed to *transient*) existence of a mathematical *object*
3. Concepts may be 'expanded'
4. Concepts may be *created*, which means: the objective ontological correlate (*idea*?) which they represent is gradually 'evoked' within a process which resembles 'creation'

Cantor's account of *conceptual dependency*

According to Cantor, concepts are mutually dependent. Within a process of 'creation' (which, in fact, resembles more a process of *derivation*), one may follow a formation procedure, which, in turn, comes in steps:

1. Posit a new sign (A)
2. A is a 'new' object
3. Find the connections of A with other signs/objects, whose consistency has already been established
4. If A is consistent with all already established X's, then A is also consistent
5. Once all properties of A have been described, then A is a new objective mathematical concept (that under which the A-objects fall)

Cantor's Objectivism

Cantor's account of concepts has some important consequences:

1. **Consistency**. The only thing we have to take care of is the *consistency* of new concepts with the old ones.
2. **Freedom**. Mathematicians are entirely free in their research (anti-metaphysicalism)
3. **Evidence**. The only evidence is *conceptual*. (however, there might be further *criteria*)

Zermelo on the Relationship between Frege and Cantor

A Red Herring: Tait on the Inconsistency of Frege's Law V

Interlude: *Sundholm* on Frege and Bolzano

[Sundholm 1999] argues that Frege *did* read Bolzano "late 1905 or early 1906".

Sundholm submits that, even if "there is no supporting evidence, in the form of even remotely Bolzanian passages, in Frege's writings prior to, say, *Grundgesetze*, Vol. II from 1903", Frege was motivated to read Bolzano after the discovery of Russell's paradox, and that Bolzano's writings very much inspired Frege's treatement of the notion of *dependence* after 1906.

In *Foundations of Geometry* II (1906) adds some remarks on the notion of dependence:

(i) If A depends on the set of true propositions Ω, the logical laws applied in taking a 'logical step' from Ω to A are not to be counted among the premises, and therefore need not occur in Ω;
(ii) *factivity*: if A depends on Ω, then both A and all the thoughts in Ω are true;
(iii) *monotonicity*: A depends on Ω if A follows from "one or several of the thoughts" in Ω;
(iv) *non-antisymmetry*: it cannot be excluded that "every thought of

Sundholm on Frege and Bolzano/Cont'd

- Even if Sundholm is right that Frege did *formulate* the notion of dependence in Bolzanian terms after 1906, that notion traces back at least to Frege's *Begriffschriftt* [1879], and to his *Grundlagen* [1884];
- even if there might be a *direct* path from Bolzano to Frege *after* 1906, our suggestion is that there was also an *indirect* (Cantorian) path, and this indirect influence is crucial in order to understand Frege's central claims about the objectivity of mathematics.

Frege on Concepts and Extensions

[Frege 1892] distinguishes between the *sense* [[*Sinn*]] and *reference* [[*Bedeutung*]] of a linguistic expression.

Frege's distinction is not limited to singular terms, but it extends to *predicates* as well. According to Frege,

- the *referent* of a predicate 'P(x)' is a *concept* [[*Begriff*]], viz. an "unsatured entity" that can be saturated by an entity of an appropriate sort (e.g. an object);
- the *sense* of a predicate is a "mode of presentation" of a (Fregean) concept.

Frege on Concepts and Extensions/2

[Frege 1892b] argues that concepts can be taken as *functions* from entities of a certain type (e.g. objects) to *truth-values* (the True or the False).

Frege famously argues that

> *The possibility of regarding the equality holding generally between values of functions as a [particular] equality, viz. an equality between value-ranges is, I think, indemonstrable; it must be taken to be a fundamental law of logic.*[1]

When restricted to concepts, Frege's *Basic Law V* states that states that F and G have the same *extension* if and only if every F is a G and vice versa:

$$\forall F \forall G \ (\text{ext}(F) = \text{ext}(G) \leftrightarrow \forall x \ (F(x) \leftrightarrow G(x))) \qquad \text{(BLV)}$$

[1] [Frege 1892b: 135]

Frege on Concepts and Extensions/3

As emphasized by [Boolos 1987], Frege employed *two* relations of instantiation between objects and concepts: on the one hand, objects *fall under* concepts; at the same time, concepts *has* (or "are in") objects as their extensions.

These two relations can obtain *simultaneously*: for example, the number one falls under the concept *identical with one*, whose number is one. These remarks have important when it comes to Frege's conception of the *order* between concepts.

Frege on Concepts and Extensions/4

On the one hand, Frege's theory of concepts resembles a theory of types: objects falls under *first-level* concepts; first-level concepts fall "within" *second-level* concepts; and so on.

On the other hand, Basic Law V allowed Frege to dispense with higher-order concepts altogether, since, for any higher-order concept C, it is sufficient to consider the *first-level* concept of being an extension of a concept falling within C. Basic Law V therefore lowers the entire Fregean hierarchy of concepts down to the level of objects.

For example, [Frege 1884] defines the cardinal number of the concept F as the extension of the *second-level* concept *equinumerous with* F, while [Frege 1893] defines the number of F as the extension of the *first-level* concept *extension of a concept equinumerous with* F.

Set-Theoretic Reductionism

Platonism

Concluding Remarks