Steel's Programme: Evidential Framework, the Core and Ultimate-L

Claudio Ternullo

Universitat de Barcelona

Barcelona Set Theory Seminar
14 October 2020

Universitat de Barcelona

UNIVERSITAT DE BARCELONA

Ontology

Universism

The axioms and language of set theory are best interpreted as referring to just *one universe* of sets, V

Ontology

Universism

The axioms and language of set theory are best interpreted as referring to just *one universe* of sets, V

Multiversism

The axioms and language of set theory are best interpreted as referring to a *plurality* of *universes* of set theory

Ontology

Universism

The axioms and language of set theory are best interpreted as referring to just *one universe* of sets, V

Multiversism

The axioms and language of set theory are best interpreted as referring to a *plurality* of *universes* of set theory

Universism and multiversism are the *ontological* incarnation of the more general *pluralism/non-pluralism* divide.

UNIVERSITAT DE BARCELONA

Ontology

Universism

The axioms and language of set theory are best interpreted as referring to just *one universe* of sets, V

Multiversism

The axioms and language of set theory are best interpreted as referring to a *plurality* of *universes* of set theory

Universism and multiversism are the *ontological* incarnation of the more general *pluralism/non-pluralism* divide.

Semantic and *proof-theoretic* forms of pluralism/non-pluralism may also be articulated.

Semantics/Theories

Semantic Pluralism/Non-pluralism

Set-theoretic truth is *unique*/splits into several *alternative* truths.

Semantics/Theories

Semantic Pluralism/Non-pluralism

Set-theoretic truth is *unique*/splits into several *alternative* truths.

Proof-Theoretic Pluralism/Non-Pluralism

There is just *one* theory/are *many* theories which describe the realm of *sets*.

Semantics/Theories

Semantic Pluralism/Non-pluralism

Set-theoretic truth is *unique*/splits into several *alternative* truths.

Proof-Theoretic Pluralism/Non-Pluralism

There is just *one* theory/are *many* theories which describe the realm of *sets*.

[Steel, 2014] instantiates proof-theoretic pluralism.

Pluralist Conceptions

- Hamkins (cf. [Hamkins, 2012]): pluralist about theories+pluralist about ontology

Pluralist Conceptions

- Hamkins (cf. [Hamkins, 2012]): pluralist about theories+pluralist about ontology
- S. Friedman et al.'s Hyperuniverse Programme (cf. [Arrigoni and Friedman, 2013], [Antos et al., 2015]): non-pluralist about theories, different degrees of pluralism about ontology (from *moderate* to *radical*)

Pluralist Conceptions

- Hamkins (cf. [Hamkins, 2012]): pluralist about theories+pluralist about ontology
- S. Friedman et al.'s Hyperuniverse Programme (cf. [Arrigoni and Friedman, 2013], [Antos et al., 2015]): non-pluralist about theories, different degrees of pluralism about ontology (from *moderate* to *radical*)
- Woodin's set-generic multiverse (cf. [Woodin, 2011a]): (moderately) pluralist about ontology, non-pluralist about theories

UNIVERSITAT DE BARCELONA

Steel's conception

- Non-pluralist about statements settled by ZFC(+Large Cardinals (LCs))

Steel's conception

- Non-pluralist about statements settled by ZFC(+Large Cardinals (LCs))
- Pluralist about 'extensions of ZFC(+LCs)'

Steel's conception

- Non-pluralist about statements settled by ZFC(+Large Cardinals (LCs))
- Pluralist about 'extensions of ZFC(+LCs)'
- Dismisses facts concerning *ontology* and *semantics* as inapplicable to the debate

Steel's conception

- Non-pluralist about statements settled by ZFC(+Large Cardinals (LCs))
- Pluralist about 'extensions of ZFC(+LCs)'
- Dismisses facts concerning *ontology* and *semantics* as inapplicable to the debate
- Views the 'multiverse' as a collection of *theories* all of which 'extend' ZFC(+LCs) in mutually incompatible ways

Steel's conception

- Non-pluralist about statements settled by ZFC(+Large Cardinals (LCs))
- Pluralist about 'extensions of ZFC(+LCs)'
- Dismisses facts concerning *ontology* and *semantics* as inapplicable to the debate
- Views the 'multiverse' as a collection of *theories* all of which 'extend' ZFC(+LCs) in mutually incompatible ways
- Reduces the issue of whether there are multiple *universes of set theory* to that of whether there are *multiple theories* (of said kind)

Evidential Basis

- LCs extend ZFC in the 'correct' way

Evidential Basis

- LCs extend ZFC in the 'correct' way
- All 'natural' theories practically 'live' in *models* (*forcing extensions* and *inner models*) of ZFC+LCs

UNIVERSITAT DE BARCELONA

Evidential Basis

- LCs extend ZFC in the 'correct' way
- All 'natural' theories practically 'live' in *models* (*forcing extensions* and *inner models*) of ZFC+LCs
- Any 'natural' theory is assumed to be either equiconsistent with ZFC or with ZFC+LCs.

Evidential Basis

- LCs extend ZFC in the 'correct' way
- All 'natural' theories practically 'live' in *models* (*forcing extensions* and *inner models*) of ZFC+LCs
- Any 'natural' theory is assumed to be either equiconsistent with ZFC or with ZFC+LCs.
- A *complete* set of axioms is needed to describe the multiverse

Evidential Basis

- LCs extend ZFC in the 'correct' way
- All 'natural' theories practically 'live' in *models* (*forcing extensions* and *inner models*) of ZFC+LCs
- Any 'natural' theory is assumed to be either equiconsistent with ZFC or with ZFC+LCs.
- A *complete* set of axioms is needed to describe the multiverse
- Multiverse axioms should include/address: (a) ZFC, (b) LCs, and (c) models which are needed to *represent* (and mutually *connect*) all natural theories (which preserve LCs).

UNIVERSITAT DE BARCELONA

Evidential Basis

- LCs extend ZFC in the 'correct' way
- All 'natural' theories practically 'live' in *models* (*forcing extensions* and *inner models*) of ZFC+LCs
- Any 'natural' theory is assumed to be either equiconsistent with ZFC or with ZFC+LCs.
- A *complete* set of axioms is needed to describe the multiverse
- Multiverse axioms should include/address: (a) ZFC, (b) LCs, and (c) models which are needed to *represent* (and mutually *connect*) all natural theories (which preserve LCs).
- As far as (c) is concerned: we just need to take into account *set-forcing* extensions of V.

Evidential Basis/Cont'd.

As regards the status of LCs, consider the following:

Evidential Basis/Cont'd.

As regards the status of LCs, consider the following:

Maximise Interpretative Power (MIP)

LCs maximise interpretive power: given any two theories T and S equiconsistent with, at least, the LCA: "there are infinitely many Woodin cardinals", then if $Con(T) \to Con(S)$, then $Th(S) \subseteq Th(T)$, where $Th(T) = \{\varphi : T \vdash \varphi\}$.

Evidential Basis/Cont'd.

As regards the status of LCs, consider the following:

Maximise Interpretative Power (MIP)

LCs maximise interpretive power: given any two theories T and S equiconsistent with, at least, the LCA: "there are infinitely many Woodin cardinals", then if $Con(T) \rightarrow Con(S)$, then $Th(S) \subseteq Th(T)$, where $Th(T) = \{\varphi : T \vdash \varphi\}$.

But note that:

UNIVERSITAT DE BARCELONA

Evidential Basis/Cont'd.

As regards the status of LCs, consider the following:

Maximise Interpretative Power (MIP)

LCs maximise interpretive power: given any two theories T and S equiconsistent with, at least, the LCA: "there are infinitely many Woodin cardinals", then if $Con(T) \rightarrow Con(S)$, then $Th(S) \subseteq Th(T)$, where $Th(T) = \{\varphi : T \vdash \varphi\}$.

But note that:

- MIP is true up to the level of Σ^1_ω-statements.

UNIVERSITAT DE BARCELONA

Evidential Basis/Cont'd.

As regards the status of LCs, consider the following:

Maximise Interpretative Power (MIP)

LCs maximise interpretive power: given any two theories T and S equiconsistent with, at least, the LCA: "there are infinitely many Woodin cardinals", then if $Con(T) \rightarrow Con(S)$, then $Th(S) \subseteq Th(T)$, where $Th(T) = \{\varphi : T \vdash \varphi\}$.

But note that:

- MIP is true up to the level of Σ^1_ω-statements.
- Status of the claim: 'all 'natural' theories extending ZFC are equiconsistent with ZFC or with ZFC+LCs' is controversial.

Evidential Basis/Cont'd.

As regards the status of LCs, consider the following:

Maximise Interpretative Power (MIP)

LCs maximise interpretive power: given any two theories T and S equiconsistent with, at least, the LCA: "there are infinitely many Woodin cardinals", then if $Con(T) \to Con(S)$, then $Th(S) \subseteq Th(T)$, where $Th(T) = \{\varphi : T \vdash \varphi\}$.

But note that:

- MIP is true up to the level of Σ^1_ω-statements.
- Status of the claim: 'all 'natural' theories extending ZFC are equiconsistent with ZFC or with ZFC+LCs' is controversial.
- It isn't clear whether Steel wants to *represent* theories or *amalgamate* them into one theory: ZFC+LCs. If the former is the case, then the argument based on MIP doesn't seem to be really required.

Evidential Basis/Cont'd.

Further problems:

Evidential Basis/Cont'd.

Further problems:

- What is a 'natural' theory?

Evidential Basis/Cont'd.

Further problems:

- What is a 'natural' theory?
- No general definition of LC is available

Evidential Basis/Cont'd.

Further problems:

- What is a 'natural' theory?
- No general definition of LC is available

These problems are addressed in [Maddy and Meadows, 2020].

Evidential Basis/Cont'd.

Further problems:

- What is a 'natural' theory?
- No general definition of LC is available

These problems are addressed in [Maddy and Meadows, 2020].

We further address them in [Bagaria and Ternullo, 2020], but I will not deal with these in this talk.

UNIVERSITAT DE BARCELONA

The Axioms

Clearly, Steel is a *pluralist* about statements not settled by ZFC+LCs (such as CH).

The Axioms

Clearly, Steel is a *pluralist* about statements not settled by ZFC+LCs (such as CH).

Goal. *Find multiverse axioms (MV), which make sense of the proof-theoretic multiverse of 'extensions' of ZFC(+LCs).*

UNIVERSITAT DE BARCELONA

The Axioms

Clearly, Steel is a *pluralist* about statements not settled by ZFC+LCs (such as CH).

Goal. *Find multiverse axioms (MV), which make sense of the proof-theoretic multiverse of 'extensions' of ZFC(+LCs).*

Multiverse language, $\mathcal{L}_{MV}$, is related to 'standard' set-theoretic language ($\mathcal{L}_{\in}$) via a 'translation function':

The Axioms

Clearly, Steel is a *pluralist* about statements not settled by ZFC+LCs (such as CH).

Goal. *Find multiverse axioms (MV), which make sense of the proof-theoretic multiverse of 'extensions' of ZFC(+LCs).*

Multiverse language, $\mathcal{L}_{MV}$, is related to 'standard' set-theoretic language ($\mathcal{L}_{\in}$) via a 'translation function':

$$t : \mathcal{L}_{MV} \longrightarrow \mathcal{L}_{\in}$$

UNIVERSITAT DE BARCELONA

The Axioms

Clearly, Steel is a *pluralist* about statements not settled by ZFC+LCs (such as CH).

Goal. *Find multiverse axioms (MV), which make sense of the proof-theoretic multiverse of 'extensions' of ZFC(+LCs).*

Multiverse language, $\mathcal{L}_{MV}$, is related to 'standard' set-theoretic language ($\mathcal{L}_{\in}$) via a 'translation function':

$$t : \mathcal{L}_{MV} \longrightarrow \mathcal{L}_{\in}$$

Note: if (Amalgamation) holds (see next slide), then (Transl) also holds ([Maddy and Meadows, 2020], Theorem 10).

MV

The language is $\mathcal{L}_{\in}$, with two sorts of variables: one for 'worlds', and one for 'sets'. The base theory T is ZFC(+LCs).

MV

The language is $\mathcal{L}_{\in}$, with two sorts of variables: one for 'worlds', and one for 'sets'. The base theory T is ZFC(+LCs).

1. (Extensionality for Worlds) If two worlds have the same sets, then they are equal. ([Maddy and Meadows, 2020], p. 26)

UNIVERSITAT DE BARCELONA

MV

The language is $\mathcal{L}_\in$, with two sorts of variables: one for 'worlds', and one for 'sets'. The base theory T is ZFC(+LCs).

1. (Extensionality for Worlds) If two worlds have the same sets, then they are equal. ([Maddy and Meadows, 2020], p. 26)
2. φ^W, for every axiom φ of T and every world W.

MV

The language is $\mathcal{L}_\in$, with two sorts of variables: one for 'worlds', and one for 'sets'. The base theory T is ZFC(+LCs).

1. (Extensionality for Worlds) If two worlds have the same sets, then they are equal. ([Maddy and Meadows, 2020], p. 26)
2. φ^W, for every axiom φ of T and every world W.
3. Every *world* is a transitive proper class. An object is a *set* if and only if it belongs to some world. All worlds have the same ordinals.

MV

The language is $\mathcal{L}_{\in}$, with two sorts of variables: one for 'worlds', and one for 'sets'. The base theory T is ZFC(+LCs).

1. (Extensionality for Worlds) If two worlds have the same sets, then they are equal. ([Maddy and Meadows, 2020], p. 26)
2. φ^W, for every axiom φ of T and every world W.
3. Every *world* is a transitive proper class. An object is a *set* if and only if it belongs to some world. All worlds have the same ordinals.
4. If W is a world and $\mathbb{P} \in W$ is a poset, then there is a world of the form $W[G]$, where G is $\mathbb{P}$-generic over W.

MV

The language is $\mathcal{L}_\in$, with two sorts of variables: one for 'worlds', and one for 'sets'. The base theory T is ZFC(+LCs).

1. (Extensionality for Worlds) If two worlds have the same sets, then they are equal. ([Maddy and Meadows, 2020], p. 26)
2. φ^W, for every axiom φ of T and every world W.
3. Every *world* is a transitive proper class. An object is a *set* if and only if it belongs to some world. All worlds have the same ordinals.
4. If W is a world and $\mathbb{P} \in W$ is a poset, then there is a world of the form $W[G]$, where G is $\mathbb{P}$-generic over W.
5. If U is a world and $U = W[G]$, where G is $\mathbb{P}$-generic over W, then W is also a world.

MV

The language is $\mathcal{L}_\in$, with two sorts of variables: one for 'worlds', and one for 'sets'. The base theory T is ZFC(+LCs).

1. (Extensionality for Worlds) If two worlds have the same sets, then they are equal. ([Maddy and Meadows, 2020], p. 26)
2. φ^W, for every axiom φ of T and every world W.
3. Every *world* is a transitive proper class. An object is a *set* if and only if it belongs to some world. All worlds have the same ordinals.
4. If W is a world and $\mathbb{P} \in W$ is a poset, then there is a world of the form $W[G]$, where G is $\mathbb{P}$-generic over W.
5. If U is a world and $U = W[G]$, where G is $\mathbb{P}$-generic over W, then W is also a world.
6. (Amalgamation) If U and W are worlds, then there are posets $\mathbb{P} \in U$ and $\mathbb{Q} \in W$, and sets G and H $\mathbb{P}$-generic and $\mathbb{Q}$-generic over U and W, respectively, such that $U[G] = W[H]$.

The Model

A natural model of Steel's MV is M^G, a variant of the set-generic multiverse.

The Model

A natural model of Steel's MV is M^G, a variant of the set-generic multiverse.

Definition (M^G)

For M a ctm of ZFC and G, $Col(\omega, < Ord^M)$-generic over M, let M^G be the set of countable models N such that for some $\alpha \in Ord^M$, $\mathbb{P} \in N$ and H $\mathbb{P}$-generic over N, $N[H] = M[G \restriction \alpha]$.[a]

[a] Based on [Steel, 2014], p. 166, and [Maddy and Meadows, 2020], p. 28.

UNIVERSITAT DE BARCELONA

The Model

A natural model of Steel's MV is M^G, a variant of the set-generic multiverse.

Definition (M^G)

For M a ctm of ZFC and G, $Col(\omega, < Ord^M)$-generic over M, let M^G be the set of countable models N such that for some $\alpha \in Ord^M$, $\mathbb{P} \in N$ and H $\mathbb{P}$-generic over N, $N[H] = M[G \restriction \alpha]$.[a]

[a] Based on [Steel, 2014], p. 166, and [Maddy and Meadows, 2020], p. 28.

Theorem

MV is complete.[a] *That is:*

$$MV \vdash \phi \leftrightarrow (\forall M) M^G \models \phi$$

where M is a ctm of ZFC.

[a] For a proof of the theorem, see [Maddy and Meadows, 2020], Theorem 8.

V and MV

Are there traces of the Universist's V in MV?

V and MV

Are there traces of the Universist's V in MV?

- *Weak Relativism*: whatever is meaningful in $\mathcal{L}_{\in}$ may preserve its meaning in $\mathcal{L}_{MV}$ (so, V *may*, potentially, be meaningful for $\mathcal{L}_{MV}$)

UNIVERSITAT DE BARCELONA

V and MV

Are there traces of the Universist's V in MV?

- *Weak Relativism*: whatever is meaningful in $\mathcal{L}_{\in}$ may preserve its meaning in $\mathcal{L}_{MV}$ (so, V *may*, potentially, be meaningful for $\mathcal{L}_{MV}$)
- *Weak Absolutism*: there is an 'individual world' which is definable in $\mathcal{L}_{MV}$ (so, V *clearly is* meaningful for $\mathcal{L}_{MV}$)

V and MV

Are there traces of the Universist's V in MV?

- *Weak Relativism*: whatever is meaningful in $\mathcal{L}_{\in}$ may preserve its meaning in $\mathcal{L}_{MV}$ (so, V *may*, potentially, be meaningful for $\mathcal{L}_{MV}$)
- *Weak Absolutism*: there is an 'individual world' which is definable in $\mathcal{L}_{MV}$ (so, V *clearly is* meaningful for $\mathcal{L}_{MV}$)

The Weak Absolutist, thus, thinks that the Universist's use of V can still be made sense of in MV through defining a special world, say $\mathcal{C}$, which is: (a) unique; (b) contained in all worlds.

V and MV

Are there traces of the Universist's V in MV?

- *Weak Relativism*: whatever is meaningful in $\mathcal{L}_{\in}$ may preserve its meaning in $\mathcal{L}_{MV}$ (so, V *may*, potentially, be meaningful for $\mathcal{L}_{MV}$)
- *Weak Absolutism*: there is an 'individual world' which is definable in $\mathcal{L}_{MV}$ (so, V *clearly is* meaningful for $\mathcal{L}_{MV}$)

The Weak Absolutist, thus, thinks that the Universist's use of V can still be made sense of in MV through defining a special world, say $\mathcal{C}$, which is: (a) unique; (b) contained in all worlds.

$\mathcal{C}$ = *the core*.

Core Universism

The *ontological pluralist*'s rendition of Weak Absolutism is:

Core Universism

The *ontological pluralist*'s rendition of Weak Absolutism is:

Core Universism

There exists a *multiplicity* of universes of set theory, all of which contain a *'preferred' universe* $\mathcal{C}$.

Core Universism

The *ontological pluralist*'s rendition of Weak Absolutism is:

Core Universism

There exists a *multiplicity* of universes of set theory, all of which contain a *'preferred' universe* $\mathcal{C}$.

By contrast, the Classic Universist holds the view that:

Core Universism

The *ontological pluralist*'s rendition of Weak Absolutism is:

Core Universism

There exists a *multiplicity* of universes of set theory, all of which contain a *'preferred' universe* $\mathcal{C}$.

By contrast, the Classic Universist holds the view that:

Classic Universism

There is just one universe of set theory, V.

Core Universism

The *ontological pluralist*'s rendition of Weak Absolutism is:

Core Universism

There exists a *multiplicity* of universes of set theory, all of which contain a *'preferred' universe* $\mathcal{C}$.

By contrast, the Classic Universist holds the view that:

Classic Universism

There is just one universe of set theory, V.

Steel's programme, alongside formulating the MV axioms, also envisages mathematically and philosophically making sense of Weak Absolutism (Core Universism) within MV.

Universitat de Barcelona

(Some) Geology

Definition (Ground)

A ground of V is a model W, such that V has been obtained from W through a (non-trivial) forcing $\mathbb{P} \in W$.

(Some) Geology

Definition (Ground)

A ground of V is a model W, such that V has been obtained from W through a (non-trivial) forcing $\mathbb{P} \in W$.

Definition (Mantle, κ-Mantle, Bedrock)

1. The mantle $\mathbb{M}$ of V is the *intersection* of all *grounds* of V.

(Some) Geology

Definition (Ground)

A ground of V is a model W, such that V has been obtained from W through a (non-trivial) forcing $\mathbb{P} \in W$.

Definition (Mantle, κ-Mantle, Bedrock)

1. The mantle $\mathbb{M}$ of V is the *intersection* of all *grounds* of V.
2. Let κ be a cardinal. The κ-mantle of V is the intersection of all grounds W of V, such that V has been obtained from W through a forcing $\mathbb{P} \in W$ of size $< \kappa$.

(Some) Geology

Definition (Ground)

A ground of V is a model W, such that V has been obtained from W through a (non-trivial) forcing $\mathbb{P} \in W$.

Definition (Mantle, κ-Mantle, Bedrock)

1. The mantle $\mathbb{M}$ of V is the *intersection* of all *grounds* of V.
2. Let κ be a cardinal. The κ-mantle of V is the intersection of all grounds W of V, such that V has been obtained from W through a forcing $\mathbb{P} \in W$ of size $< \kappa$.
3. The bedrock is the *smallest* ground of V.

(Some) Geology

Definition (Ground)

A ground of V is a model W, such that V has been obtained from W through a (non-trivial) forcing $\mathbb{P} \in W$.

Definition (Mantle, κ-Mantle, Bedrock)

1. The mantle $\mathbb{M}$ of V is the *intersection* of all *grounds* of V.
2. Let κ be a cardinal. The κ-mantle of V is the intersection of all grounds W of V, such that V has been obtained from W through a forcing $\mathbb{P} \in W$ of size $< \kappa$.
3. The bedrock is the *smallest* ground of V.

Axiom (Ground Axiom (GA), [Reitz, 2007])

V has not been obtained through a non-trivial forcing over any ground W (V has no proper grounds).

Prelude: the Outer Core of *ZFC*

Take some model V of ZFC, and let $\mathbb{M}$ be its mantle. Suppose $\mathbb{M} \models \neg GA$. Then, one may produce the 'mantle of the mantle' $\mathbb{M}^{\mathbb{M}}$. In general, one may recursively define: $\mathbb{M}^{\alpha+1} = \mathbb{M}(\mathbb{M}^{\alpha})$.

Prelude: the Outer Core of *ZFC*

Take some model V of ZFC, and let $\mathbb{M}$ be its mantle. Suppose $\mathbb{M} \models \neg GA$. Then, one may produce the 'mantle of the mantle' $\mathbb{M}^{\mathbb{M}}$. In general, one may recursively define: $\mathbb{M}^{\alpha+1} = \mathbb{M}(\mathbb{M}^{\alpha})$.

Definition (Outer Core)

Suppose that, for $\alpha, \beta \in Ord$, $\mathbb{M}^{\alpha} = \mathbb{M}^{\beta}$, for all $\beta > \alpha$. Then, $\mathbb{M}^{\alpha}$ is the *outer core* of V.

UNIVERSITAT DE BARCELONA

Prelude: the Outer Core of *ZFC*

Take some model V of ZFC, and let $\mathbb{M}$ be its mantle. Suppose $\mathbb{M} \models \neg GA$. Then, one may produce the 'mantle of the mantle' $\mathbb{M}^{\mathbb{M}}$. In general, one may recursively define: $\mathbb{M}^{\alpha+1} = \mathbb{M}(\mathbb{M}^{\alpha})$.

Definition (Outer Core)

Suppose that, for $\alpha, \beta \in Ord$, $\mathbb{M}^{\alpha} = \mathbb{M}^{\beta}$, for all $\beta > \alpha$. Then, $\mathbb{M}^{\alpha}$ is the *outer core* of V.

Theorem ([Reitz and Williams, 2019])

It is consistent with ZFC that, for all $\alpha \in Ord$, the outer core of V is $\mathbb{M}^{\alpha}$.

The Core of ZFC+LCs

Theorem ([Usuba, 2019])

Assume there exists an extendible cardinal κ. Then, the κ-mantle of V is its smallest ground.

The Core of ZFC+LCs

Theorem ([Usuba, 2019])

Assume there exists an extendible cardinal κ. Then, the κ-mantle of V is its smallest ground.

The theorem above may, then, be used to prove that:

The Core of ZFC+LCs

Theorem ([Usuba, 2019])

Assume there exists an extendible cardinal κ. Then, the κ-mantle of V is its smallest ground.

The theorem above may, then, be used to prove that:

Theorem ([Bagaria and Ternullo, 2020], draft version)

Let T be ZFC+ "there exists a proper class of extendible cardinals". Then, the MV_T axioms imply that the multiverse has a core, which is the mantle (and a ground) of every world in the multiverse.

The Core of ZFC+LCs

Theorem ([Usuba, 2019])

Assume there exists an extendible cardinal κ. Then, the κ-mantle of V is its smallest ground.

The theorem above may, then, be used to prove that:

Theorem ([Bagaria and Ternullo, 2020], draft version)

Let T be ZFC+ "there exists a proper class of extendible cardinals". Then, the MV_T axioms imply that the multiverse has a core, which is the mantle (and a ground) of every world in the multiverse.

Note: the existence of a *class*, not just *one*, extendible is needed to preserve LCs in MV-worlds.

A Threshold?

It would be natural, at this point, to wonder whether the large-cardinal assumption of Usuba's theorem can be 'weakened'. The following result provides a (negative) response to this question:

A Threshold?

It would be natural, at this point, to wonder whether the large-cardinal assumption of Usuba's theorem can be 'weakened'. The following result provides a (negative) response to this question:

Theorem ([Bagaria and Ternullo, 2020], draft version)

Assume $T = ZFC +$ "there exists a proper class of supercompacts". Then, MV_T does not prove that the multiverse has a core.

A Threshold?

It would be natural, at this point, to wonder whether the large-cardinal assumption of Usuba's theorem can be 'weakened'. The following result provides a (negative) response to this question:

Theorem ([Bagaria and Ternullo, 2020], draft version)

Assume T=ZFC+"there exists a proper class of supercompacts". Then, MV_T does not prove that the multiverse has a core.

Moreover, this is very speculative, but it seems improbable that consistency strength weaker LCs would be sufficient to prove the existence of the core.

Indeterminacy of the Core

The core is a highly *indeterminate* object, as shown by:

Indeterminacy of the Core

The core is a highly *indeterminate* object, as shown by:

Theorem (Reitz)

Every model V of ZFC(+LCs) has a class-forcing extension $V[G]$ such that $V[G] \models GA$.

Indeterminacy of the Core

The core is a highly *indeterminate* object, as shown by:

Theorem (Reitz)

Every model V of ZFC(+LCs) has a class-forcing extension $V[G]$ such that $V[G] \models GA$.

So, in particular, there may be extensions of *any* world of MV, which satisfy GA, and, respectively, CH or $\neg$CH.

Indeterminacy of the Core

The core is a highly *indeterminate* object, as shown by:

Theorem (Reitz)

Every model V of ZFC(+LCs) has a class-forcing extension $V[G]$ such that $V[G] \models GA$.

So, in particular, there may be extensions of *any* world of MV, which satisfy GA, and, respectively, CH or $\neg$CH.

[Bagaria and Poveda, 2020] extends Reitz's results and techniques, so as to further nail down the indeterminacy of the core. E.g., consider:

Indeterminacy of the Core

The core is a highly *indeterminate* object, as shown by:

Theorem (Reitz)

Every model V of ZFC(+LCs) has a class-forcing extension $V[G]$ such that $V[G] \models GA$.

So, in particular, there may be extensions of *any* world of MV, which satisfy GA, and, respectively, CH or $\neg$CH.

[Bagaria and Poveda, 2020] extends Reitz's results and techniques, so as to further nail down the indeterminacy of the core. E.g., consider:

Theorem ([Bagaria and Ternullo, 2020], draft version)

Let φ be a Σ_2-statement (with parameters) that can be forced by set-forcing. Assume there is a proper class of extendible cardinals. Then in some class-forcing extension of V that preserves extendible cardinals the statement φ holds in the core of its Steel's multiverse.

The Core and Forcing Axioms

The indeterminacy of the core also extends to Π_2 sentences (with parameters); in particular, the core may, or may not, satisfy *forcing axioms*.

The Core and Forcing Axioms

The indeterminacy of the core also extends to Π_2 sentences (with parameters); in particular, the core may, or may not, satisfy *forcing axioms*.

Theorem ([Bagaria and Ternullo, 2020], draft version)

If T =ZFC+ "there exists a proper class of extendibles" is consistent, then so is T+that the core of MV_T satisfies $MA_{\aleph_1}$.

UNIVERSITAT DE BARCELONA

The Core and Forcing Axioms

The indeterminacy of the core also extends to Π_2 sentences (with parameters); in particular, the core may, or may not, satisfy *forcing axioms*.

Theorem ([Bagaria and Ternullo, 2020], draft version)

If T =ZFC+ "there exists a proper class of extendibles" is consistent, then so is T+that the core of MV_T satisfies $MA_{\aleph_1}$.

Theorem ([Bagaria and Ternullo, 2020], draft version)

If ZFC+the existence of a supercompact cardinal is consistent, then so is MM^{++} plus the GA.

The Core and Forcing Axioms

The indeterminacy of the core also extends to Π_2 sentences (with parameters); in particular, the core may, or may not, satisfy *forcing axioms*.

Theorem ([Bagaria and Ternullo, 2020], draft version)

If T =ZFC+ "there exists a proper class of extendibles" is consistent, then so is T+that the core of MV_T satisfies $MA_{\aleph_1}$.

Theorem ([Bagaria and Ternullo, 2020], draft version)

If ZFC+the existence of a supercompact cardinal is consistent, then so is MM^{++} plus the GA.

And by recent results in [Asperò and Schindler, 2019], the theorem above also extends to Woodin's $(*)$ axiom.

The Ultimate-L Conjecture

Definition (V =Ultimate-L (Woodin))

The axiom V =Ultimate-L asserts:

1. There is a proper class of Woodin cardinals, *and*
2. For each Σ_2 sentence φ, if φ holds in V, then there is a universally-Baire set $A \subseteq \mathbb{R}$ such that

$$HOD^{L(A,\mathbb{R})} \models \varphi.$$

The Ultimate-L Conjecture

Definition (V =Ultimate-L (Woodin))

The axiom V =Ultimate-L asserts:

1. There is a proper class of Woodin cardinals, *and*
2. For each Σ_2 sentence φ, if φ holds in V, then there is a universally-Baire set $A \subseteq \mathbb{R}$ such that

$$HOD^{L(A,\mathbb{R})} \models \varphi.$$

The definition above is, itself, true in Ultimate-L: that is, a canonical (weak extender) model for *supercompacts*.

The Ultimate-L Conjecture

Definition (V =Ultimate-L (Woodin))

The axiom V =Ultimate-L asserts:

1. There is a proper class of Woodin cardinals, *and*
2. For each Σ_2 sentence φ, if φ holds in V, then there is a universally-Baire set $A \subseteq \mathbb{R}$ such that

$$HOD^{L(A,\mathbb{R})} \models \varphi.$$

The definition above is, itself, true in Ultimate-L: that is, a canonical (weak extender) model for *supercompacts*.

Conjecture (Ultimate-L conjecture ([Woodin, 2017]))

Ultimate-L exists.

Ultimate-L and the Core

Woodin had already shown that, if one assumes V=Ultimate-L, then one has that:

Ultimate-L and the Core

Woodin had already shown that, if one assumes V=Ultimate-L, then one has that:

Theorem ([Woodin, 2017], Theorem 7.28)

V is the minimum universe (core) of the set-generic multiverse.

Ultimate-L and the Core

Woodin had already shown that, if one assumes V=Ultimate-L, then one has that:

Theorem ([Woodin, 2017], Theorem 7.28)

V is the minimum universe (core) of the set-generic multiverse.

However, the result below shows that the core need not be Ultimate-L.

Ultimate-L and the Core

Woodin had already shown that, if one assumes V=Ultimate-L, then one has that:

Theorem ([Woodin, 2017], Theorem 7.28)

V is the minimum universe (core) of the set-generic multiverse.

However, the result below shows that the core need not be Ultimate-L.

Theorem ([Bagaria and Ternullo, 2020], draft version)

Let T be ZFC+ "There exists a proper class of extendible cardinals". If T + "V = Ultimate-L" is consistent, then so is MV_T + "$\mathcal{C} \models V$=Ultimate-L", and MV_T + "$\mathcal{C} \not\models V$ =Ultimate-L".

Steel's MV and Ultimate-L as the Core

MV_T, with T =ZFC+LCs, and LCs=proper class of extendibles, implies that there is a core.

Steel's MV and Ultimate-L as the Core

MV_T, with T =ZFC+LCs, and LCs=proper class of extendibles, implies that there is a core.

Notice that the core is a *definable* world, and it can be proved that, if the multiverse has a definable world, then that world is *unique*. So, in the presence of LCs=proper class of extendibles, that world is the core.

Steel's MV and Ultimate-L as the Core

MV_T, with T =ZFC+LCs, and LCs=proper class of extendibles, implies that there is a core.

Notice that the core is a *definable* world, and it can be proved that, if the multiverse has a definable world, then that world is *unique*. So, in the presence of LCs=proper class of extendibles, that world is the core.

Now, the (Core) Universist might think that this fact encourages her to adopt $V = \mathcal{C}$ as one further axiom.

Steel's MV and Ultimate-L as the Core

MV_T, with T =ZFC+LCs, and LCs=proper class of extendibles, implies that there is a core.

Notice that the core is a *definable* world, and it can be proved that, if the multiverse has a definable world, then that world is *unique*. So, in the presence of LCs=proper class of extendibles, that world is the core.

Now, the (Core) Universist might think that this fact encourages her to adopt $V = \mathcal{C}$ as one further axiom.

Can we strengthen T in such a way as to have a *direct* implication? Consider the following two cases:

Steel's MV and Ultimate-L as the Core

MV_T, with T =ZFC+LCs, and LCs=proper class of extendibles, implies that there is a core.

Notice that the core is a *definable* world, and it can be proved that, if the multiverse has a definable world, then that world is *unique*. So, in the presence of LCs=proper class of extendibles, that world is the core.

Now, the (Core) Universist might think that this fact encourages her to adopt $V = \mathcal{C}$ as one further axiom.

Can we strengthen T in such a way as to have a *direct* implication? Consider the following two cases:

1. Take T=ZFC+LCs+V =Ultimate-L. The theory MV_T would be inconsistent (Axiom 4 would no longer hold).

Steel's MV and Ultimate-L as the Core

MV_T, with T =ZFC+LCs, and LCs=proper class of extendibles, implies that there is a core.

Notice that the core is a *definable* world, and it can be proved that, if the multiverse has a definable world, then that world is *unique*. So, in the presence of LCs=proper class of extendibles, that world is the core.

Now, the (Core) Universist might think that this fact encourages her to adopt $V = \mathcal{C}$ as one further axiom.

Can we strengthen T in such a way as to have a *direct* implication? Consider the following two cases:

1. Take T=ZFC+LCs+V =Ultimate-L. The theory MV_T would be inconsistent (Axiom 4 would no longer hold).
2. Take T=ZFC+LCs+"$\mathcal{C} \models V$=Ultimate-L". MV_T would imply that, if one adopts $V = \mathcal{C}$, then the core is Ultimate-L, but V still need not be the core.

More on Ultimate-L and MV

Now, take T, in MV_T, to be T=ZFC+LCs+"$\mathcal{C} \models V$=Ultimate-L".

As said, the Universist may now have some reasons to shift to ZFC+LCs+V=Ultimate-L. The theory implies $V = \mathcal{C}$, so, through using the 'translation function', one will have that:

UNIVERSITAT DE BARCELONA

More on Ultimate-L and MV

Now, take T, in MV_T, to be T=ZFC+LCs+"$\mathcal{C} \models V$=Ultimate-L".

As said, the Universist may now have some reasons to shift to ZFC+LCs+V=Ultimate-L. The theory implies $V = \mathcal{C}$, so, through using the 'translation function', one will have that:

- CH$\leftrightarrow$ $t(CH^{Ultimate-L})$

More on Ultimate-L and MV

Now, take T, in MV_T, to be T=ZFC+LCs+"$\mathcal{C} \models V$=Ultimate-L".

As said, the Universist may now have some reasons to shift to ZFC+LCs+V=Ultimate-L. The theory implies $V = \mathcal{C}$, so, through using the 'translation function', one will have that:

- CH$\leftrightarrow t(CH^{Ultimate-L})$
- For all undecidable ϕ, $\phi \leftrightarrow t(\phi^{Ultimate-L})$

UNIVERSITAT DE BARCELONA

More on Ultimate-L and MV

Now, take T, in MV_T, to be T=ZFC+LCs+"$\mathcal{C} \models V$=Ultimate-L".

As said, the Universist may now have some reasons to shift to ZFC+LCs+V=Ultimate-L. The theory implies $V = \mathcal{C}$, so, through using the 'translation function', one will have that:

- CH$\leftrightarrow$ $t(CH^{Ultimate-L})$
- For all undecidable ϕ, $\phi \leftrightarrow t(\phi^{Ultimate-L})$

However, it wouldn't be clear how the MV supporter of "$\mathcal{C} \models V$ =Ultimate-L" would differentiate himself from the Classic Universist supporter of V=Ultimate-L.

More on Ultimate-L and MV

Now, take T, in MV_T, to be T=ZFC+LCs+"$\mathcal{C} \models V$=Ultimate-L".

As said, the Universist may now have some reasons to shift to ZFC+LCs+V=Ultimate-L. The theory implies $V = \mathcal{C}$, so, through using the 'translation function', one will have that:

- CH$\leftrightarrow t(CH^{Ultimate-L})$
- For all undecidable ϕ, $\phi \leftrightarrow t(\phi^{Ultimate-L})$

However, it wouldn't be clear how the MV supporter of "$\mathcal{C} \models V$ =Ultimate-L" would differentiate himself from the Classic Universist supporter of V=Ultimate-L.

The arguments mentioned by [Steel, 2014], p. 169 (that it implies $V = \mathcal{C}$, provides a fine structure theory for $\mathcal{C}$, is compatible with all LCs) are precisely the same as the ones invoked by the Classic Universist V=Ultimate-L supporter.

More on Ultimate-L and MV

Now, take T, in MV_T, to be T=ZFC+LCs+"$\mathcal{C} \models V$=Ultimate-L".

As said, the Universist may now have some reasons to shift to ZFC+LCs+V=Ultimate-L. The theory implies $V = \mathcal{C}$, so, through using the 'translation function', one will have that:

- CH$\leftrightarrow t(CH^{Ultimate-L})$
- For all undecidable ϕ, $\phi \leftrightarrow t(\phi^{Ultimate-L})$

However, it wouldn't be clear how the MV supporter of "$\mathcal{C} \models V$ =Ultimate-L" would differentiate himself from the Classic Universist supporter of V=Ultimate-L.

The arguments mentioned by [Steel, 2014], p. 169 (that it implies $V = \mathcal{C}$, provides a fine structure theory for $\mathcal{C}$, is compatible with all LCs) are precisely the same as the ones invoked by the Classic Universist V=Ultimate-L supporter.

Are there any other arguments on offer?

UNIVERSITAT DE BARCELONA

The Progression

MV_T and the core for different choices of T.

The Progression

MV_T and the core for different choices of T.

Theory	Existence	Determinacy	Suggests $V = \mathcal{C}$
ZFC	No	/	/
ZFC+LCs	Yes	No	Yes?
ZFC+LCs+V=Ult-L	**Inconsistent!**		
ZFC+LCs+"$\mathcal{C} \models V$=Ultimate-L"	Yes	Yes	Yes?
...	...	...	...

The Core Hypothesis. A Summary

- There exist multiple theories, all of which may be expressed in 'worlds' of MV

The Core Hypothesis. A Summary

- There exist multiple theories, all of which may be expressed in ‘worlds’ of MV
- If there are sufficiently strong LCs, then MV implies the existence of a ‘core universe’

The Core Hypothesis. A Summary

- There exist multiple theories, all of which may be expressed in 'worlds' of MV
- If there are sufficiently strong LCs, then MV implies the existence of a 'core universe'
- Ascending through *consistency strength* of LCs, however, does not help fix the features of the core

The Core Hypothesis. A Summary

- There exist multiple theories, all of which may be expressed in 'worlds' of MV
- If there are sufficiently strong LCs, then MV implies the existence of a 'core universe'
- Ascending through *consistency strength* of LCs, however, does not help fix the features of the core
- The adoption, on the Universist's part, of $V = \mathcal{C}$, in particular, of V=Ultimate-L is 'suggested', but not 'required' by MV_T, where T =ZFC+LCs

The Core Hypothesis. A Summary

- There exist multiple theories, all of which may be expressed in 'worlds' of MV
- If there are sufficiently strong LCs, then MV implies the existence of a 'core universe'
- Ascending through *consistency strength* of LCs, however, does not help fix the features of the core
- The adoption, on the Universist's part, of $V = \mathcal{C}$, in particular, of V=Ultimate-L is 'suggested', but not 'required' by MV_T, where T =ZFC+LCs
- The only consistent strengthening of T in MV_T, through adding "$\mathcal{C} \models$Ultimate-L" only implies that, *if* V really is the core, then it is Ultimate-L; but one should justify "$\mathcal{C} \models$Ultimate-L" from the point of view of MV, anyway

UNIVERSITAT DE BARCELONA

A Cosmological Principle..

In [Woodin, 2011b], the clash between a (generically) *formalist* and *non-formalist* viewpoint about set theory is rendered in terms of the dialectic between a Skeptic and a Set-Theorist.

A Cosmological Principle..

In [Woodin, 2011b], the clash between a (generically) *formalist* and *non-formalist* viewpoint about set theory is rendered in terms of the dialectic between a Skeptic and a Set-Theorist.

- *Skeptic*: set-theoretic theorems are just finitary truths expressed by *proofs*.

A Cosmological Principle..

In [Woodin, 2011b], the clash between a (generically) *formalist* and *non-formalist* viewpoint about set theory is rendered in terms of the dialectic between a Skeptic and a Set-Theorist.

- *Skeptic*: set-theoretic theorems are just finitary truths expressed by *proofs*.
- *Set-theorist*: set-theoretic theorems capture truths about an *existing* realm of mathematical objects.

A Cosmological Principle..

In [Woodin, 2011b], the clash between a (generically) *formalist* and *non-formalist* viewpoint about set theory is rendered in terms of the dialectic between a Skeptic and a Set-Theorist.

- *Skeptic*: set-theoretic theorems are just finitary truths expressed by *proofs*.
- *Set-theorist*: set-theoretic theorems capture truths about an *existing* realm of mathematical objects.

Set-Theorist's Cosmological Principle

LCs, for which there are *inner models*, are consistent; the prediction of their consistency is true because the axioms are *true*.

A Cosmological Principle..

In [Woodin, 2011b], the clash between a (generically) *formalist* and *non-formalist* viewpoint about set theory is rendered in terms of the dialectic between a Skeptic and a Set-Theorist.

- *Skeptic*: set-theoretic theorems are just finitary truths expressed by *proofs*.
- *Set-theorist*: set-theoretic theorems capture truths about an *existing* realm of mathematical objects.

Set-Theorist's Cosmological Principle

LCs, for which there are *inner models*, are consistent; the prediction of their consistency is true because the axioms are *true*.

So, inner models would precisely encapsulate the truth of LCs. Ultimate-L, if it exists, would encapsulate the truth of *all* LCs.

..and how to use it within Steel's MV

By the Cosmological Principle, if one believes ZFC+LCs, as Steel does, then one should also believe that:

UNIVERSITAT DE BARCELONA

..and how to use it within Steel's MV

By the Cosmological Principle, if one believes ZFC+LCs, as Steel does, then one should also believe that:

- LCs are best interpreted as 'expressing features' of a *core universe* which contains *all of them*.

..and how to use it within Steel's MV

By the Cosmological Principle, if one believes ZFC+LCs, as Steel does, then one should also believe that:

- LCs are best interpreted as 'expressing features' of a *core universe* which contains *all of them*.

Based on the fact above, the *MV* supporter could then contend that the best candidate as the core of MV_T, where T =ZFC+LCs, is Ultimate-L.

..and how to use it within Steel's MV

By the Cosmological Principle, if one believes ZFC+LCs, as Steel does, then one should also believe that:

- LCs are best interpreted as 'expressing features' of a *core universe* which contains *all of them*.

Based on the fact above, the *MV* supporter could then contend that the best candidate as the core of MV_T, where T =ZFC+LCs, is Ultimate-L.

Two problems:

UNIVERSITAT DE BARCELONA

..and how to use it within Steel's MV

By the Cosmological Principle, if one believes ZFC+LCs, as Steel does, then one should also believe that:

- LCs are best interpreted as 'expressing features' of a *core universe* which contains *all of them*.

Based on the fact above, the *MV* supporter could then contend that the best candidate as the core of MV_T, where T =ZFC+LCs, is Ultimate-L.

Two problems:

- The evidential resources invoked by the argument (truth, consistency predictions, etc.) do not seem to be available to the *MV* supporter

UNIVERSITAT DE BARCELONA

..and how to use it within Steel's MV

By the Cosmological Principle, if one believes ZFC+LCs, as Steel does, then one should also believe that:

- LCs are best interpreted as 'expressing features' of a *core universe* which contains *all of them*.

Based on the fact above, the *MV* supporter could then contend that the best candidate as the core of MV_T, where T =ZFC+LCs, is Ultimate-L.

Two problems:

- The evidential resources invoked by the argument (truth, consistency predictions, etc.) do not seem to be available to the *MV* supporter
- At bottom, there would still be little difference between the *Universist V =Ultimate-L supporter*, and the *Multiversist MV_T supporter (with T =ZFC+LCs+"$\mathcal{C} \models$Ultimate-L")*

Further arguments

The *MV* supporter might view "$\mathcal{C} \models V$=Ultimate-L" as *extrinsically* justified by the fact that V=Ultimate-L has an incredibly strong degree of canonicity.

Further arguments

The *MV* supporter might view "$\mathcal{C} \models V$=Ultimate-L" as *extrinsically* justified by the fact that V=Ultimate-L has an incredibly strong degree of canonicity.

The 'niceness' and successfulness of V=Ultimate-L are unquestionable. However, notice that, say, "$\mathcal{C} \models MM^{++}$" (or, for that matter, "$\mathcal{C} \models FA$", where FA is a forcing axiom), might be justified on the same grounds (although, arguably, there are no *canonical* models for FAs).

Further arguments

The *MV* supporter might view "$\mathcal{C} \models V$=Ultimate-L" as *extrinsically* justified by the fact that V=Ultimate-L has an incredibly strong degree of canonicity.

The 'niceness' and successfulness of V=Ultimate-L are unquestionable. However, notice that, say, "$\mathcal{C} \models MM^{++}$" (or, for that matter, "$\mathcal{C} \models FA$", where FA is a forcing axiom), might be justified on the same grounds (although, arguably, there are no *canonical* models for FAs).

Moreover, it isn't clear that such an extrinsic argument would be in line with *MV*'s evidential framework (first few slides).

Concluding Scenarios for Steel's Programme (and Core Universism)

1. Stronger LCs will imply more *determinacy* of the core

Concluding Scenarios for Steel's Programme (and Core Universism)

1. Stronger LCs will imply more *determinacy* of the core
2. The value of the 'core hypothesis' may just consist in allowing us to exhibit the theoretical interplay between *multiverse* and *universe thinking* and provide a (meta-)mathematical characterisation of such an interplay

UNIVERSITAT DE BARCELONA

Concluding Scenarios for Steel's Programme (and Core Universism)

1. Stronger LCs will imply more *determinacy* of the core
2. The value of the 'core hypothesis' may just consist in allowing us to exhibit the theoretical interplay between *multiverse* and *universe thinking* and provide a (meta-)mathematical characterisation of such an interplay
3. The issue of the existence of a 'preferred universe' cannot be solved proof-theoretically (the issue is 'non-linguistic'), but should, rather, make use of further *evidential* resources

End Slide

Thanks for your attention!

UNIVERSITAT DE BARCELONA

Antos, C., Friedman, S.-D., Honzik, R., and Ternullo, C. (2015).
Multiverse Conceptions in Set Theory.
Synthese, 192(8):2463–2488.

Arrigoni, T. and Friedman, S. (2013).
The Hyperuniverse Program.
Bulletin of Symbolic Logic, 19(1):77–96.

Asperò, D. and Schindler, R. (2019).
MM^{++} implies $(*)$.
Pre-print.

Bagaria, J. and Poveda, A. (2020).
More on the Preservation of Large Cardinals by Class Forcing.
Pre-print.

Bagaria, J. and Ternullo, C. (2020).
Steel's Programme: Evidential Framework, the Core and Ultimate-L.

Pre-print.

Hamkins, J. D. (2012).
The Set-Theoretic Multiverse.
Review of Symbolic Logic, 5(3):416–449.

Maddy, P. and Meadows, T. (2020).
A Reconstruction of Steel's Multiverse Project.
Bulletin of Symbolic Logic, forthcoming.

Reitz, J. (2007).
The Ground Axiom.
The Journal of Symbolic Logic, 72(4):1299–1317.

Reitz, J. and Williams, K. (2019).
Inner mantles and iterated HOD.
Mathematical Logic Quarterly, pages 1–13.

Steel, J. (2014).
Gödel's Program.

In Kennedy, J., editor, *Interpreting Gödel. Critical Essays*, pages 153–179. Cambridge University Press, Cambridge.

Usuba, T. (2019).
Extendible cardinals and the mantle.
Arch. Math. Logic, 58:71–75.

Woodin, W. H. (2011a).
The Continuum Hypothesis, the Generic-Multiverse of Sets, and the Ω-Conjecture.
In Kennedy, J. and Kossak, R., editors, *Set Theory, Arithmetic, and Foundations of Mathematics: Theorems, Philosophies*, pages 13–42. Cambridge University Press, Cambridge.

Woodin, W. H. (2011b).
The Transfinite Universe.
In M., B., C.H., P., D.S., S., and H., P., editors, *Horizons of Truth. Kurt Gödel and the Foundations of Mathematics*, pages 449–74. Cambridge University Press, Cambridge.

Woodin, W. H. (2017).

In Search for Ultimate-L. The 19th Midrasha Mathematicae Lectures.
Bulletin of Symbolic Logic, 23(1):1–109.

UNIVERSITAT DE BARCELONA