Steel's Multiverse Theory and the Ultimate Universe of Sets

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The Questions

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(Sketch of an) Answer

Define:

- Notion of multiverse
- Notion of 'preferred' ('ultimate') universe
- Notion of 'selection'

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In contrast with other versions of the set-generic multiverse, Steel's set-generic multiverse can be axiomatised and the axioms are *complete*.

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1. (Extensionality for Worlds) If two worlds have the same sets, then they are equal. ([Maddy and Meadows, 2020], p. 26)
2. φ^W, for every axiom φ of T and every world W.
3. Every *world* is a transitive proper class. An object is a *set* if and only if it belongs to some world. All worlds have the same ordinals.
4. If W is a world and $\mathbb{P} \in W$ is a poset, then there is a world of the form $W[G]$, where G is $\mathbb{P}$-generic over W.
5. If U is a world and $U = W[G]$, where G is $\mathbb{P}$-generic over W, then W is also a world.
6. (Amalgamation) If U and W are worlds, then there are posets $\mathbb{P} \in U$ and $\mathbb{Q} \in W$, and sets G and H $\mathbb{P}$-generic and $\mathbb{Q}$-generic over U and W, respectively, such that $U[G] = W[H]$.

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Definition (M^G)

For M a ctm of ZFC and G, $Col(\omega, < Ord^M)$-generic over M, let M^G be the set of countable models N such that for some $\alpha \in Ord^M$, $\mathbb{P} \in N$ and H $\mathbb{P}$-generic over N, $N[H] = M[G \restriction \alpha]$.[1]

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Theorem (Completeness of the MV axioms)

MV is complete.[2] *That is:*

$$MV \vdash \phi \leftrightarrow (\forall M) M^G \models \phi$$

where M is a ctm of ZFC.

[2] For a proof of the theorem, see [Maddy and Meadows, 2020], Theorem 8.

MV: The Translation Function

The multiverse language, $\mathcal{L}_{MV}$, is a *sublanguage* of $\mathcal{L}_{\in}$, as *all* statements expressible in $\mathcal{L}_{MV}$ are also expressible in $\mathcal{L}_{\in}$ via the 'translation function':

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In particular, one has that:

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for all sentences ϕ of the multiverse language. $t(\phi)$ just says 'ϕ is true in some (equivalently all) multiverse(s) obtained from me' ([Steel, 2014], p. 166)

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(Transl) is central to Steel's purposes, insofar as it, potentially, allows one to *translate* facts about a 'target' universe expressed in $\mathcal{L}_{MV}$ to $\mathcal{L}_{\in}$ (see next slide).

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Steel's Programme

Steel's Programme (partly based on [Steel, 2014])

Let MV_T be the multiverse theory whose base theory is T=ZFC+LCA, and which contains a *definable* universe (which is *unique* and is called the *core*). Then the following hold:

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3. MV_T motivates/justifies the claim that the core is Ultimate-L (the 'preferred' universe is Ultimate-L)

Main Strategy

Suppose MV_T proves $\Phi =$ 'there is a *universe* of MV_T with *unique* properties'. Now, $t(\Phi) \in \mathcal{L}_\in$ will imply (or, at least, suggest) that V *is such a universe*.

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Thus, Steel's Programme, in a sense, reduces to the issue of whether Core Universism can be persuasively articulated and defended.

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If not, it might still be used to disconfirm the view that CH is a legitimate question ([Steel, 2014], p. 168).

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Axiom (Ground Axiom (GA), [Reitz, 2007])

V has not been obtained through a non-trivial forcing over any ground W (V has no proper grounds).

Prelude: the Outer Core of *ZFC*

Take some model V of ZFC, and let $\mathbb{M}$ be its mantle. Suppose $\mathbb{M} \models \neg GA$. Then, one may produce the 'mantle of the mantle' $\mathbb{M}^{\mathbb{M}}$. In general, one may recursively define: $\mathbb{M}^{\alpha+1} = \mathbb{M}(\mathbb{M}^{\alpha})$.

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Suppose that, for $\alpha, \beta \in$ *Ord*, $\mathbb{M}^{\alpha} = \mathbb{M}^{\beta}$, for all $\beta > \alpha$. Then, $\mathbb{M}^{\alpha}$ is the *outer core* of V.

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Theorem ([Reitz and Williams, 2019])

For all $\alpha \in Ord$, it is consistent with ZFC that the outer core of V is $\mathbb{M}^{\alpha}$.

The Core of ZFC+LCA

Definition (Extendible cardinal)

A cardinal κ is η-extendible if there is a ζ and elementary embedding $j : V_{\kappa+\eta} \to V_\zeta$, with critical point κ and such that $j(\kappa) > \eta$; κ is extendible if it is η-extendible for all $\eta > 0$.

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Theorem ([Bagaria and Ternullo, 2020], draft version)

Let T be ZFC+*"there exists a proper class of extendible cardinals". Then,* MV_T *proves that the multiverse has a core, which is the mantle (and a ground) of every world in the multiverse.*[4]

[4] Note: the existence of a *class*, not just *one*, extendible is needed to preserve LCs in *MV*-worlds.

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Theorem ([Bagaria and Ternullo, 2020], draft version)

Let φ be a Σ_2-statement (with parameters) that can be forced by set-forcing. Assume there is a proper class of extendible cardinals. Then in some class-forcing extension of V that preserves extendible cardinals the statement φ holds in the core of its Steel multiverse.

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And by recent results in [Asperò and Schindler, 2019], the theorem above also extends to Woodin's $(*)$ axiom.

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Conjecture (Ultimate-L conjecture ([Woodin, 2017]))

Ultimate-L exists.

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Theorem ([Bagaria and Ternullo, 2020], draft version)

Let T be ZFC+"There exists a proper class of extendible cardinals". If T + "V = Ultimate-L" is consistent, then so is MV_T + "$\mathcal{C} \models V$=Ultimate-L", and MV_T + "$\mathcal{C} \not\models V$=Ultimate-$L$".

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As to (3): there are no traces in $\mathcal{L}_{MV}$ of the fact that the core *has to be* Ultimate-L; as a matter of fact, we have seen that the core *need not be* Ultimate-L.

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   MV_T would imply that, if one adopts $V = \mathcal{C}$, then the core is Ultimate-L, but V still need not be the core.

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- *Unbiasedness* of MV is violated.

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Theory	Existence	Determinacy	Suggests $V = \mathcal{C}$
ZFC	No	/	/
ZFC+LCA	Yes	No	Yes?
ZFC+LCA+V=Ult-L	**Inconsistent!**		
ZFC+LCA+"$\mathcal{C} \models V$=Ultimate-L"	Yes	Yes	Yes?
...	...	...	...

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- The only consistent strengthening of T in MV_T, through adding "$\mathcal{C} \models$Ultimate-L" only implies that, *if* V really is the core, then it is Ultimate-L (but Main Strategy is inapplicable)

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So, inner models express the truth of LCs. Ultimate-L, if it exists, would express the truth of *all* LCs compatible with AC.

..and how to use it within Steel's MV

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- The evidential resources invoked by the argument (truth, consistency predictions, etc.) do not seem to be available to the MV supporter
- At bottom, there would still be little difference between the *Universist* V =*Ultimate-L supporter*, and the *Multiversist* MV_T *supporter* (with T =ZFC+LCs+"$\mathcal{C} \models$Ultimate-L")

Further arguments

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Moreover, it isn't clear that such an extrinsic argument would be in line with MV's evidential framework.

Concluding Scenarios for Steel's Programme (and Core Universism)

1. Stronger LCs will imply more *determinacy* of the core (in the presence of the failure of the Main Strategy, this could still be seen as further motivation to endorse $V = \mathcal{C}$)

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2. The value of the 'core hypothesis' may just consist in allowing us to exhibit the theoretical interplay between *multiverse* and *universe thinking* and provide a (meta-)mathematical characterisation of such an interplay
3. The issue of the existence of a 'preferred universe' cannot be solved proof-theoretically (the issue is 'non-linguistic'), but should, rather, make use of further *evidential* resources

End Slide

Thanks for your attention!

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