Varieties of Set-Theoretic Pluralism

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Preliminaries

- Set theory is the *foundation* of mathematics, which means that all mathematical entities and concepts may be translated to set-theoretic entities and concepts
- The (most commonly used) axioms of set theory are the **ZFC** axioms
- By Gödel's Incompleteness Theorems, **ZFC** must be *incomplete*
- So, there are statements that **ZFC** is not able to *decide* (that is, it does not either *prove* or *disprove* them). Among such statements, there are many which are *genuine* set-theoretic statements (unlike Con(**ZFC**), these have a *genuine* set-theoretic *content*)
- Let a statement A be *determinate* if, in **ZFC**, there is either a proof of A, or a proof of $\neg A$; otherwise A is *indeterminate*
- Many set-theoretic statements, thus, are *indeterminate*

Independence and Pluralism

Independence Phenomenon

Let M and N be models of **ZFC**. Let A be a set-theoretic statement. If $M \models A$ and $N \models \neg A$, then, if **ZFC** is consistent, it does not prove A or $\neg A$ (so, A is *independent* from **ZFC**).

Multiverse Phenomenon

There are *universes* of **ZFC** where an indeterminate statement A is true, and others where it is false.

'Quick Fix' Strategy

Let A be indeterminate. Then, the theory **ZFC**$+A$ fixes the truth-value of A. Problems:

- A is not on an equal conceptual footing with the other **ZFC** axioms.
- **ZFC**$+A$ is also *incomplete*.
- **ZFC**$+A$ and **ZFC**$+\neg A$ might both be *interesting* theories (which of A and $\neg A$ has a stronger claim to be seen as a *new* axiom?)

The Concept of Set

One further reason for not accepting the Quick Fix Strategy is that **ZFC** is motivated by the:

Iterative Concept of Set (ICS)

The sets are the elements of the *well-founded, cumulative hierarchy* V (the universe of sets).

Now, suppose A is *indeterminate*. **ZFC**$+A$ might not be motivated by the ICS.

However, alternatively, **ZFC**$+A$ might be motivated by its consequences ('extrinsically', cf. [Gödel, 1947]).

A 'less quick' fix consists in:

- Finding an axiom Φ (possibly, a statement which has the same status as **ZFC**), such that Φ proves A, where A is indeterminate.
- Thus, **ZFC**$+\Phi$ is an *intrinsically* justified extension of **ZFC**.

Problem: after years of multiple investigations on *new* axioms, no such *new* statement has been found which is as uncontroversial as **ZFC**. We might just have been unlucky, or ICS is also *hopelessly* indeterminate.

Set-Theoretic Pluralism

We have two, equally unpalatable, scenarios ahead of us:

Proof-Theoretic Pluralism

Any theory T which extends **ZFC** is as good as any other, as there is no *set-theoretic reality*, and no fixed concept of *set-theoretic truth*.

Universism

Set theory is the first-order theory of V, the universe of sets. 'Truth in V' is a fully *meaningful*, and, in principle, *completable*, notion.

A compromise solution is represented by some version of *set-theoretic pluralism*:

Pluralism

Set theory is the theory of a collection of *universes*, each of which instantiates one *concept of set* and *collection of set-theoretic truths*.

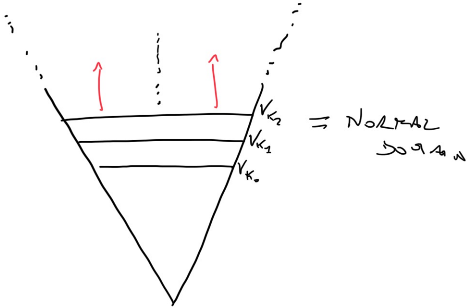
Four Main Multiverse Conceptions

1. Zermelo's Quasi-Universism (cf. [Zermelo, 1930])
2. The Set-Generic Multiverse ([Steel, 2014], [Woodin, 2011])
3. The V-logic Multiverse ([Antos et al., 2015], [Antos et al., 2020])
4. The Radical Multiverse ([Hamkins, 2012])

In turn, these may grouped into three main categories:

1. **universism with a (partially) incomplete** V, that is, height extensions of V (Zermelo)
2. **multiverse of ZFC (plus, possible, Large Cardinals)** both height and width extensions (outer models) of V+inner models (Friedman et al., Woodin, Steel, Väänänen)
3. **two-dimensional pluralism**: proof-theoretic pluralism plus *all* conceivable models (Hamkins)

Zermelo's 'Natural' Models

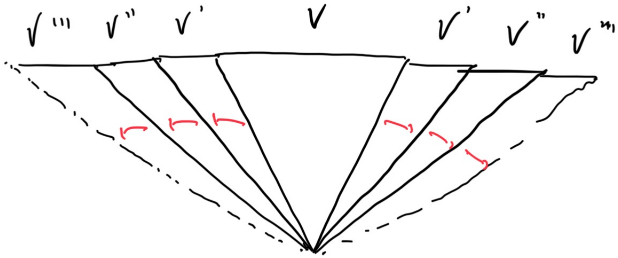


Base Theory is

ZFC_2

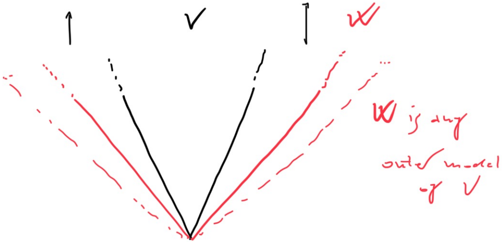
(ZFC with second-order versions of replacement and separation))

Set-Generic Multiverse



$V', V'', V''', \ldots$ ARE ALL SET-GENERIC EXTENSIONS OF V

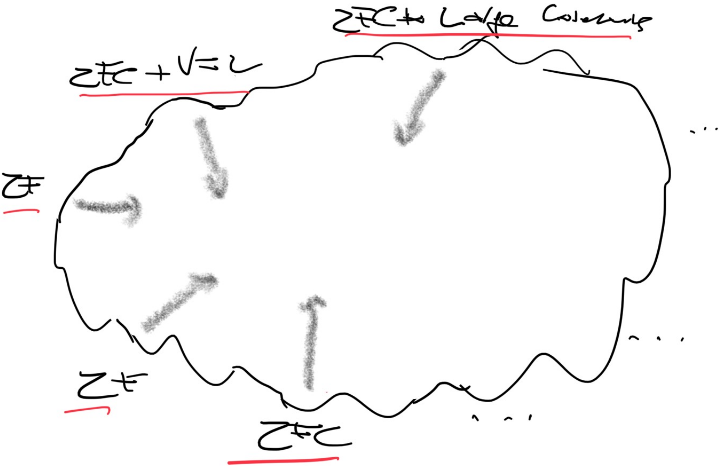
V-logic Multiverse



W is an OUTER MODEL of V:

1. SET-GENERIC EXTENSION
2. CLASS-GENERIC EXTENSION
3. ...

Radical Multiverse



Summary

Conception	**Theories**	**Models**	**Axioms**
Zermelo's Quasi-Universism	$\mathbf{ZFC}_2$	Normal Domains	No
Set-Generic Multiverse	**ZFC**(+LCs)	Set-Generic Extensions	Yes/No
V-logic Multiverse	**ZFC**	All	Yes
Radical Multiverse	All	All	No
...	...	...	...

A Bunch of Philosophical Questions

1. *Foundations*: in what sense would a *multiverse theory* be best suited to be a foundation of mathematics?
2. *Truth*: in what sense and to what extent is set-theoretic (mathematical) truth *split*?
3. *Relativism*: do theories of the multiverse support a *relativist* view about *set-theoretic truth* and *concepts*?
4. *Skepticism*: do multiverse theories support a skeptical view of set-theoretic (mathematical) truth?
5. *Logical pluralism*: do multiverse theories entail any form of logical pluralism?

The ‘Foundation’ Issue

- *(Set-theoretic) naturalism* is the view that all philosophical issues relating to set theory are answerable by attending to the details of ‘set-theoretic practice’
- The view has been formulated by Maddy in several works ([Maddy, 1996], [Maddy, 1997])
- Our best theory of sets is thus the one which suits best intra-set-theoretic goals. So:

> *The choice between a universe approach and a multiverse approach is justified to the extent that it facilitates our set-theoretic goals. The universe advocate finds good reasons for his view in the many jobs that it does so well, at which point the challenge is turned back to the multiverse advocate: given that we could work with inner models and forcing extensions from within the simple confines of V, as described by our best universe theory, what mathematical motivation is there to move to a more complex multiverse theory? ([Maddy, 2017], p. 316)*

Maddy concludes on a skeptical note: there is presently no cogent reason why one should shift from ‘**ZFC**+V’ to the ‘multiverse of (some theory T of sets)’ as one’s *foundational* framework for mathematics.

Naturalised Multiversism

In [Ternullo, 2019] I have argued that the Maddian's intra-set-theoretic (and foundational) goals may all be achieved *also* when interpreting set theory as a theory of a plurality of universes:

- (Experience) Current set theory mostly deals with *models* of set theory.
- (Unification) Theories may be *amalgamated*: any *mathematical* theory may be seen as living in a *forcing extension* or *inner model* of the theory: **ZFC**+Large Cardinal Axioms ([Steel, 2014])
- (Discovery) The multiverse has proved successful in isolating principles which reduce *set-theoretic incompleteness*
- (Conceptual Elucidation) The 'multiverse phenomenon' helps one elucidate different ways to construe the concept of set

My conclusion was that, even from the perspective of a *naturalistic* approach to mathematics (and set theory), set-theoretic pluralism is sanctioned by a huge amount of both *experiential* and *conceptual* data.

Hamkins on the Multiverse

The background idea of the multiverse, of course, is that there should be a large collection of universes, each a model of (some kind of) set theory. There seems to be no reason to restrict inclusion only to ZFC models, as we can include models of weaker theories ZF, ZF^-, KP, and so on, perhaps even down to second-order number theory, as this is set-theoretic in a sense. At the same time, there is no reason to consider all universes in the multiverse equally, and we may simply be more interested in the parts of the multiverse consisting of universes satisfying very strong theories, such as ZFC plus large cardinals. ([Hamkins, 2012], p. 436)

Before describing these principles, however, let me briefly remark on the issue of formalism. Of course we do not expect to be able to express universe existence principles in the first-order language of set theory or even in the second-order language of Gödel-Bernays or Kelly-Morse set theory, since the entire point of the multiverse perspective is that there may be other universes outside a given one. ([Hamkins, 2012], p. 436)

Is The Radical Multiverse Conception Tenable?

By the radical view, the multiverse is the collection of *all* universes of *all* (fragments of) theories **T** of sets.

Problems ([Koellner, 2013], [Barton, 2016], [Button and Walsh, 2018]):

- (Articulation)
    - In order to articulate the multiverse of **ZFC** you need to work in **ZFC**+Con(**ZFC**), but, then, in order to articulate the multiverse of **ZFC**+Con(**ZFC**), you have to work in Con(**ZFC**+Con(**ZFC**)) and so on, indefinitely;
    - Suppose you have managed to articulate the multiverse of **ZFC**, working in a background theory **ZFC**+Con(**ZFC**); in the multiverse, there is a universe M such that $M \models \neg\text{Con}(\textbf{ZFC})$, which contradicts what is (externally) stated by **ZFC**+Con(**ZFC**)
- (Threshold) There is no *lower* bound, in such a multiverse, for where the indeterminacy phenomenon starts (e.g., there is no principled reason why one should view Con(**PA**) as *determinate*)
- (Characterisability) Forcing extensions $V[G]$ of the universe V may not be seen as 'existent' objects by any sense of the word

Two Strategies for the Radical Multiversist

1. *Full-Blown Perspectivism*: by this interpretation, 'there exist *universes* U, V, W' is conceptually different from: 'assuming the consistency of T, then T has *models* (has access to) $U, V, W \ldots$'
2. *Plentiful Ontology*: by this interpretation, the theory of the multiverse is *equivalent* to some other theory (which, possibly, uses some alternative logic) positing the existence of such *abstract objects* as *universes*

Cont'd (1.)

By (1), typical multiverse statements should just be taken to express *metatheoretic* facts such as:

'If **ZFC** is *consistent*, then **ZFC** has models U, V, W'

which do not commit the radical multiversist to the (absolute) *existence* of the multiverse.

For instance, take Hamkins':

Countability Axiom (CA). Every universe V is *countable* from the perspective of another, better, universe W.

By strategy (1), this statement should just be taken to assert that:

'If **ZFC** is consistent, then it has *countable transitive models* which satisfy **CA**.'

Cont'd (2).

(2) is based on, among other things, a:

Principle of Plenitude (cf. [Balaguer, 1995]). Any mathematical entity which *can* be consistently conceived of as existing, *does exist*

By (2)., we have that universes should be seen as objects *encoding collections of properties* ([Zalta, 1983], [Linsky and Zalta, 1995]).

Compare:

- Ba: a has the property B.
- aB: a encodes B

Now, take a model of **ZFC**, say, M. Now, using Linsky and Zalta's *object theory*, a typical member of the multiverse could be described as follows:

> 'The (unique) object o such that o *encodes* all properties P such that **ZFC** $\vdash P$ plus other properties (such as, $\mathfrak{c} = \aleph_2$)'.

Thus, the radical multiverse may be seen as a collection of *special* platonic objects which differ from each other by the *properties* they *encode*.

So, (2) implies that *multiverse-theoretic* statements be *translated* into *object-theoretic statements*.

Skepticism

Strategy (1.) may be seen as exemplifying skepticism about set-theoretic truth (*Skolemism*: set-theoretic concepts are *context-sensitive*, and/or do not presuppose an *absolute* background *concept of set*).

However, it isn't clear that Skolem's thinking is necessarily *skeptical*: certainly it is not skeptical about our ability to capture the truth of *finitary*, *metamathematical* statements about the multiverse.

On the other hand, Zalta and Linsky's *object theory* is a theory of special metaphysical entities, which, in principle, are axiomatically describable in a wholly *complete* way.

Consequently, it cannot be seen as exemplifying a form of *skepticism* about, at least, our ability to grasp *truths* about those *objects*.

However, it could be seen as supporting a skeptical view about the face-value *meaning* and *interpretation* of set-theoretic statements (thus, the *referential indeterminacy/vagueness* of set-theoretic *axioms* and *theorems*).

End

Thanks for your attention!

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